21st International Symposium on
Mathematical Theory of Networks and Systems
July 7-11, 2014. Groningen, The Netherlands

Symmetries of Multiagent Systems and Formation Stability

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Abstract— **This work is part of a larger research effort investigating the role of symmetries in control of large-scale cyber-physical-systems (CPS). Most prior efforts have considered discrete symmetries, and the work in this paper reflects our initial efforts directed toward investigating continuous symmetries. Specifically, we consider a formation control problem, limited for now to only two agents, and determine all the point transformation symmetries associated with that system. We also determine a reduced description of the dynamics of the system particularly important for formation control and show stability of the system in those coordinates.**

I. INTRODUCTION

This paper considers the multi-agent formation control problem characterized by two important symmetry features. First, by "formation" we mean a desired relative configuration among mobile agents that only depends upon the relative positioning of the agents, and not on their absolute position. Second, the individual agents are symmetric in the sense that they are identical and interact with their neighbors in the same way. This work involves the initial steps to bring together the benefits of both types of symmetries for formation control.

Since the formation only depends on the relative configuration of the agents, a successful formation can be translated and rotated in the full space of the system and hence is characterized by an SE(n) symmetry. If the control law only depends on the relative configuration of the agents and the individual agent dynamics are independent of absolute position, then the dynamics of the system will also be invariant with respect to SE(n). Hence, the formation stability problem is more easily and naturally considered on some quotient space defined by factoring out the SE(n) symmetry.

Such a symmetry actually complicates the formation stability analysis if it is considered in the full space for the system because there are an infinite number of valid formations, so stability can not be formulated in terms of an equilibrium. Hence LaSalle's Invariance Principle must be used instead of a Lyapunov approach. While LaSalle's Principle has many attractive attributes, one limitation is the need to show the existence of a compact invariant set for the system, at least for the manner in which the Principle is normally stated (see [1]). In the case where *only* the relative configuration is considered and a non-zero steady-state velocity can result, such a set does not exist because the agents may converge to the desired relative positions, but with a non-zero steady-state overall formation velocity. Even with damping on the absolute velocity terms, if one is to allow arbitrary initial conditions, showing the existence of this set can be problematic. In contrast, if the dynamics are expressed on a quotient manifold associated with the symmetry some simplifications result because it typically becomes stability of an equilibrium point.

We are certainly not the first to recognize the appeal of the symmetry in this problem. Books on differential equations and continuous symmetries include [2], [3]. In [4] the problem of defining unique formations (up to a symmetry) using a graph-theoretic formulation is addressed and local stability of formations is shown using LaSalle's Principle via a definition of a neighborhood of a formation. In [5] flocking convergence is established by the definition of a moving frame at the center of mass of the vehicles, which, along with assumptions on the control law, establishes the necessary invariant compact set to use LaSalle's Principle. In [6] (and some related papers), flocking convergence is established by defining a Lyapunov function that depends only on the relative positioning of the agents. All of these references hint at and make use of aspects of reduction to a quotient space (especially notions such as "center of mass coordinates" and a V that depends only on relative configurations), but none of them fully explore it.

A closely related reference is [7], where the formation control problem for a group of robots is abstracted in a manner where the control law is formulated in terms of a low-dimensional trivial fiber bundle type structure composed of group and shape components. Then individual robot control laws are designed for convergence to the desired shape. A second closely-related set of publications is from [8], [9] involving the use of potential functions for formation control and, especially in [9], making use of classical reduction theory from mechanics for coordinated control of rigid bodies. In contrast, this work is not restricted to mechanical systems, but rather allows for more generic dynamics.

Some of our prior work considers symmetric control systems wherein the system is composed of repeated instances of identical components [10], [11], [12], [13], which is, to some degree, a nonlinear extension of [14], [15], *i,e,*, discrete symmetries. The focus of [10] is to exploit the discrete symmetric structure of the system for stability independent of the number of components in the system. The focus of [11] is to extend such results to approximately symmetric systems. The main utility of such stability results is scalability. If the system is stable for a given number of components, it is then guaranteed for a larger system composed of the same type of components with a similar interconnection structure.

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ISBN: 978-90-367-6321-9

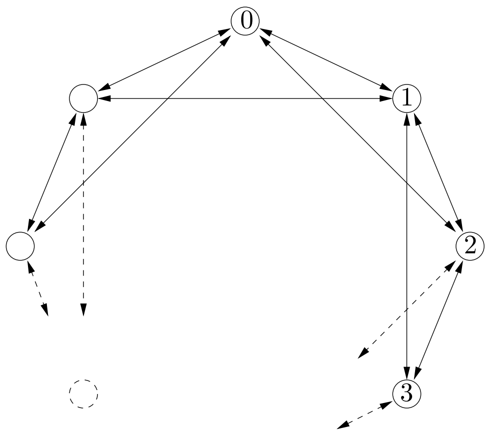

Fig. 1. System with discrete symmetric structure.

II. SYMMETRIC SYSTEMS

Consider the system illustrated in Figure 1 where each node in the graph represents an agent and each edge represents interactions among the agents.

As detailed in [10], if the inputs and outputs for each agent are related with sufficient regularity, then the system has *periodic interconnections*. If the dynamics of each agent are the same and the system has periodic interconnections, then the system is a *symmetric system* and an equivalence class of symmetric systems can be defined for different numbers of agents. If the system has a positive-definite Lyapunov function with a negative definite derivative and the Lyapunov function can be appropriately decomposed into a sum of terms corresponding to each agent, then it is shown that stability of only one system in the equivalence class implies stability for all members of the equivalence class.

As a specific example, let the formation be defined by a distance metric between an agent i and the agents in its neighborhood, denoted by $\mathcal{N}_i$. For example, in Figure 1, $\mathcal{N}_i$ contains the two agents in the counter-clockwise direction from i and the two agents in the clock-wise direction from i. Consider the controlled dynamics of each agent to be

$$\ddot{x}_i = u_{x,i} = -\dot{x}_i - \sum_{j\in\mathcal{N}_i}(x_i - x_j)d_{ij}$$
$$\ddot{y}_i = u_{y,i} = -\dot{y}_i - \sum_{j\in\mathcal{N}_i}(y_i - y_j)d_{ij}$$

with

$$d_{ij} = (x_i - x_j)^2 + (y_i - y_j)^2 - \hat{d}_{ij},$$

where $\hat{d}_{ij}$ is the square of the desired distance between agents i and j.

For this system, consider

$$V_i = \frac{1}{2}(\dot{x}_i^2 + \dot{y}_i^2) + \frac{1}{8}\sum_{j\in\mathcal{N}_i} d_{ij}^2,$$

with $V = \sum_{i=1}^{N} V_i$. This then gives

$$\dot{V} = -\sum_{i=1}^{N}(\dot{x}_i^2 + \dot{y}_i^2).$$

This is negative semi-definite, which of course means that we cannot infer asymptotic stability from Lyapunov's Theorem. One might infer notions of asymptotic stability properties from LaSalle's Principle. However, it is not straight-forward to define an invariant compact set containing all of the desired formations, as the initial conditions play a significant role in determining where the formation will be in space. In [10] this was addressed in the examples by adding a term to the control attracting the formation to the origin, which allowed for the identification of an invariant compact set which led to the use of LaSalle's Principle. Then, using the discrete symmetry, scaling of $\mathcal{O}(0)$ was obtained where the order is with respect to the number of agents in the system.

In the present efforts, we want to project the dynamics onto the quotient space defined by the symmetries of the system to allow for similar results without the need for a term attracting the formation to the origin. The goal of the present work is to make use of the symmetries present in the problem to rigorously factor out the dependence of the dynamics and stability analysis on the explicit position of the agents, and instead only depend on the relative position between them. To show some of the details of this approach, we consider a simple two-agent model using the same control approach as above.

For the two agent system, there is only one distance term and control laws simplify to

$$\begin{aligned}
\ddot{x}_1 &= \omega_1 = -\dot{x}_1 - d_{12}(x_1 - x_2)\\
\ddot{y}_1 &= \omega_2 = -\dot{y}_1 - d_{12}(y_1 - y_2)\\
\ddot{x}_2 &= \omega_3 = -\dot{x}_2 - d_{21}(x_2 - x_1)\\
\ddot{y}_2 &= \omega_4 = -\dot{y}_2 - d_{21}(y_2 - y_1)
\end{aligned} \tag{1}$$

with

$$d_{12} = (x_1 - x_2)^2 + (y_1 - y_2)^2 - \hat{d}_{12} = d_{21},$$

where the ω_i terms are defined for use subsequently.

In order to factor out the symmetries present in a system, one must first be able to find and represent the symmetries. It is possible to represent Lie symmetries by an infinitesimal point transformation in terms of at least one parameter. For example, the rotational symmetry in the plane can be represented by the point transformation

$$\tilde{x}(x, y; \varepsilon) = x\cos\varepsilon - y\sin\varepsilon$$
$$\tilde{y}(x, y; \varepsilon) = x\sin\varepsilon + y\cos\varepsilon,$$

where ε can be thought of as the angle of rotation. A function, $f(x, y) = 0$ has a rotational symmetry if $f(x, y) = f(\tilde{x}, \tilde{y}) = 0$ for the point transformation defined above.

It is also possible to represent the symmetry as a linear operator, called a generator and denoted by X. The generator can be found from the point transformation or a general form can be assumed to find all of the symmetries of a system. The generator can be defined by terms in the Taylor series expansion of the point transformation about $\varepsilon = 0$. Consider

$$\begin{aligned}\tilde{x}(x, y; \varepsilon) &= \tilde{x}(x, y; 0) + \left.\frac{\partial \tilde{x}}{\partial \varepsilon}\right|_{\varepsilon=0} (\varepsilon - 0) + \cdots \\ &= x + \varepsilon \eta_1(x, y) + \cdots \\ &= x + \varepsilon X x + \cdots \\ \tilde{y}(x, y; \varepsilon) &= \tilde{y}(x, y; 0) + \left.\frac{\partial \tilde{y}}{\partial \varepsilon}\right|_{\varepsilon=0} (\varepsilon - 0) + \cdots \\ &= y + \varepsilon \eta_2(x, y) + \cdots \\ &= y + \varepsilon X y + \cdots ,\end{aligned}$$

which gives

$$X = \eta_1(x, y)\frac{\partial}{\partial x} + \eta_2(x, y)\frac{\partial}{\partial y}.$$

Note that is also possible to have a transformation in the independent variable. The rest of this paper is limited to point transformations in the dependent variables or a shift in one independent variable. More general transformations in the independent variables are possible, which introduce some complexities. These do not apply to the system we are considering, and therefore not included in this paper.

The check for whether a function, $f(x, y) = 0$, has the symmetry defined by a generator, X, is simply $Xf = 0$, which is satisfied if the gradient of the function is orthogonal to the tangent to the action of the point transformation. For a rotational symmetry, the generator is

$$X = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}.$$

In order to apply the generator to differential equations, we must first prolong it to include derivatives. This is done by taking the Taylor series expansion of the expressions for transforming the derivatives. For an independent variable t, the expression for the prolongation of the generator is

$$\begin{aligned}X = \eta_1(x, y)\frac{\partial}{\partial x} + \eta_2(x, y)\frac{\partial}{\partial y} + \frac{d}{dt}(\eta_1(x, y))\frac{\partial}{\partial x'} \\ + \frac{d}{dt}(\eta_2(x, y))\frac{\partial}{\partial y'} + \cdots ,\end{aligned}$$

provided that there is either no transformation in the independent variable or only a shift by a constant scalar. It is possible to prolong the generator when this is not the case, however, for brevity it is not included. For further details, see [3].

The prolongation of the rotational symmetry is then

$$X = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} - y'\frac{\partial}{\partial x'} + x'\frac{\partial}{\partial y'} - y''\frac{\partial}{\partial x''} + x''\frac{\partial}{\partial y''},$$

where prime denotes the derivative with respect to time, t.

A differential equation, $H = 0$, has a symmetry defined by a generator, X, if $XH = 0$. Note that for systems of differential equations, $H_1 = 0, H_2 = 0$, it is possible that $XH_1 = H_2$, which equals zero since H_2 equals zero by definition.

One way to find the symmetries of a differential equation is by "solving for X." This is done systematically when the differential equation is expressed as a linear operator. This relationship is found by the first integrals, f of a differential equation and is

$$Af = \left(\frac{\partial}{\partial t} + y'\frac{\partial}{\partial y} + \cdots + y^{(n)}\frac{\partial}{\partial y^{(n-1)}}\right) f = 0.$$

Note that a system of differential equations can be represented by a single linear operator. Then by use of the skew symmetric Lie commutator, $[M, N] = MN - NM$, the symmetries of a differential equation can be found by

$$[X, A] = \lambda(t, y, y', \ldots, y^{(n-1)})A,$$

which is equivalent to

$$X\omega = \eta^{(n)},$$

where $H = y^{(n)} - \omega = 0$. Recall that this ω is defined for the example system in Equation (1). For a system of equations, this expression becomes $X\omega_a = A\eta_a^{(n)}$, with a as an index.

This equation was used to find all of the symmetries for the two-agent model defined above. The symmetries for this system are

$$\begin{aligned}X_1 &= \frac{\partial}{\partial t} \\ X_2 &= e^{-t}(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}) \\ X_3 &= e^{-t}(\frac{\partial}{\partial y_1} + \frac{\partial}{\partial y_2}) \\ X_4 &= \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} \\ X_5 &= \frac{\partial}{\partial y_1} + \frac{\partial}{\partial y_2} \\ X_6 &= -y_1\frac{\partial}{\partial x_1} + x_1\frac{\partial}{\partial y_1} - y_2\frac{\partial}{\partial x_2} + x_2\frac{\partial}{\partial y_2} \\ X_7 &= (y_1 + y_2)(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}) \\ X_8 &= (x_1 + x_2)(\frac{\partial}{\partial y_1} + \frac{\partial}{\partial y_2}) \\ X_9 &= (x_1 + x_2)(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}) \\ X_{10} &= (y_1 + y_2)(\frac{\partial}{\partial y_1} + \frac{\partial}{\partial y_2}).\end{aligned}$$

The generators X_1, X_4, and X_5 correspond to a translation in time and the x- and y-directions, respectively. Planar rotation is represented by X_6.

It is possible to express the dynamics of the two-agent model in a reduced set of coordinates that have the same symmetries. An obvious choice for the first coordinate is a distance metric, d_{12}. The derivative of the distance is also invariant to the group actions as is a third variable that is a function of the distance and its derivative. Hence, we take

$$\begin{aligned}q_1 &= d_{12} = (x_1 - x_2)^2 + (y_1 - y_2)^2 - \hat{d}_{12} \\ q_2 &= (x_1 - x_2)(\dot{x}_1 - \dot{x}_2) + (y_1 - y_2)(\dot{y}_1 - \dot{y}_2) \\ q_3 &= (\dot{x}_1 - \dot{x}_2)^2 + (\dot{y}_1 - \dot{y}_2)^2 + d_{12}^2,\end{aligned}$$

and the dynamics are given by

$$\begin{aligned} \dot{q_1} &= 2q_2 \\ \dot{q_2} &= -2\hat{d}_{12}q_1 - q_2 + q_3 - 3q_1^2 \\ \dot{q_3} &= -2q_3 + 2q_1^2. \end{aligned}$$

In order to consider stability of this reduced system, consider the candidate Lyapunov function

$$V = \frac{1}{2}\begin{bmatrix} q_1 & q_2 \end{bmatrix}\begin{bmatrix} 2\hat{d}_{12} & \frac{1}{4} \\ \frac{1}{4} & 2 \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + q_3^2,$$

which gives

$$\begin{aligned} \dot{V} = &\begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix}\begin{bmatrix} -\frac{\hat{d}_{12}}{2} & -\frac{3}{40} & \frac{1}{8} \\ -\frac{3}{40} & -\frac{3}{2} & 1 \\ \frac{1}{8} & 1 & -2 \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} \\ &- \frac{3}{4}q_1^3 + 2q_1^4 - 6q_1^2q_2 - 2(q_1^2 + q_3)^2. \end{aligned}$$

Using

$$\begin{aligned} &\begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix}\begin{bmatrix} -\frac{\hat{d}_{12}}{2} & -\frac{3}{40} & \frac{1}{8} \\ -\frac{3}{40} & -\frac{3}{2} & 1 \\ \frac{1}{8} & 1 & -2 \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} \\ &\qquad \le \lambda_{max}(q_1^2 + q_2^2 + q_3^2) \end{aligned}$$

it is possible to define a domain that proves that the origin is stable. For illustrative purposes, a value of $\hat{d}_{12} = 1$ will be used, however, it is possible to show stability for values larger than 0.153 with this Lyapunov function. Note that $\hat{d}_{12}$ must be greater than 0.03125 for the Lyapunov function to be positive definite. This then gives

$$\begin{aligned} \dot{V} \le &- 0.48(q_1^2 + q_2^2 + q_3^2) - 2(q_1^2 + q_3)^2 \\ &- \frac{3}{4}q_1^3 + 2q_1^4 - 6q_1^2q_2 \end{aligned}$$

which is negative semi-definite for all values of q_i and negative definite for $-0.136 \le q_1 \le 0.172$.

A simulation illustrates both the stability of the dynamics and validity of the reduced dynamics. In Figure 2, the distance metric d_{12} is computed two different ways. The blue line corresponds to solving the system using the original (full) dynamics, while the dashed red line corresponds to the reduced dynamics.

III. CONCLUSIONS

In this paper we presented a way to find all of the Lie continuous symmetries of a system. An example was given for a two-agent system with the formation control law based on a distance metric. The reduced dynamics for the two-agent system were presented that have the same set of symmetries as the original full dynamics. Reduced dynamics for continuous symmetries are beneficial for formation control, beyond the reduction in dimension, as the stability analysis simplifies due to several relative equilibria are reduced to the origin. Lyapunov stability analysis was performed on the reduced system and the results confirmed with a simulation. The simulation confirmed that the reduced dynamics and the original full dynamics can be solved to

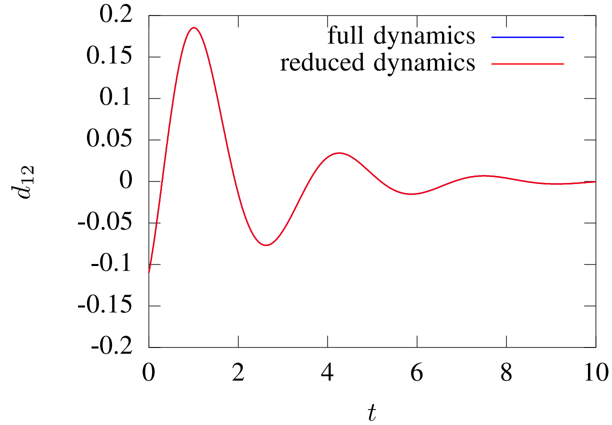

Fig. 2. Distance computation comparing full and reduced dynamics.

produce the same distance metric response. Further work will focus on extending the reduced system stability analysis for N agents and combining it with prior work done on discrete symmetries.

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