Formation Stability for Multiagent Systems with Continuous and Discrete Symmetries

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***Abstract*— This work considers the formation control problem for mobile autonomous agents. The dynamics of the controlled system will be characterized by an SE(n) symmetry in the case where the definition of the "formation," and consequently the control laws, depend only on the relative position of the agents, not their absolute position. This, of course, leads to a natural reduction problem. We also desire to bring into the problem formulation the fact that multiagent formation control problems are typically also characterized by an additional symmetry, one that is discrete and arises from the fact that each agent may be identical and interact with its neighbors in the same manner. This work aims to bring together the benefits of reduction arising from both types of symmetries.**

I. INTRODUCTION

This paper considers the multi-agent formation control problem characterized by two important symmetry features. First, by "formation" we mean a desired relative configuration among mobile agents that only depends upon the relative positioning of the agents, and not at all on their absolute position. Second, the individual agents are symmetric in the sense that they are identical (or at least diffeomorphically related) and interact with their neighbors in the same way. This work involves the initial steps to bring together the benefits of both types of symmetries in the formation control and analysis problem.

A. Continuous (Lie) Symmetry

Because the formation only depends on the relative configuration of the agents, a successful formation can be translated and rotated in the full space of the system and hence is characterized by an SE(n) symmetry. If the control law only depends on the relative configuration of the agents and the individual agent dynamics are independent of absolute position, then the dynamics of the system will be invariant with respect to an SE(n) action. Hence, the formation stability problem is more easily and naturally considered on the quotient manifold defined by "factoring out" the SE(n) symmetry.

Such a symmetry actually complicates formation stability analysis if it is considered in the full space for the system, as opposed to the quotient space. This is primarily because there are an infinite number of possible valid formations and so stability can not be formulated in terms of an equilibrium. Hence LaSalle's Invariance Principle must be used instead of a Lyapunov approach. While LaSalle's Principle has many attractive attributes, one limitation is the need to show the existence of a compact invariant set for the system, at least for the manner in which the Principle is normally stated (see [1]). In the case where *only* the relative configuration is considered and a non-zero steady-state velocity can result, such a set does not exist because the agents may converge to the desired relative positions, but with a non-zero steady-state overall formation velocity. Even with damping on the absolute velocity terms, if one is to allow arbitrary initial conditions, showing the existence of this set can be problematic. In contrast, if the dynamics are expressed on the quotient manifold associated with the symmetry some simplifications result because it typically becomes stability of an equilibrium point.

The main drawback to the reduction to a quotient space approach we are considering here is that while the existence of the quotient space and well-defined dynamics thereon are guaranteed with the right conditions on the group action [2], any particularly simple representation is not constructively given and the search for the "best" coordinates on the quotient manifold may be difficult. Fortunately, for the problem at hand there are natural invariants to guide this search, including quantities such as the desired distances among the agents, the formation linear and angular velocities, etc.

We are certainly not the first to recognize the appeal of the symmetry in this problem. In [3] the problem of definition unique formations (up to a symmetry) using a graph-theoretic formulation is addressed and local stability of formations is shown using LaSalle's Principle via a definition of a neighborhood of a formation. In [4] flocking convergence is established by the definition of a moving frame at the center of mass of the vehicles, which, along with assumptions on the control law establishes the necessary invariant compact set to use LaSalle's Principle. In [5] (and some related papers), flocking convergence is established by defining a Lyapunov function that depends only on the relative positioning of the agents and noting that the time derivative of it is negative in the space of relative positions using the full dynamics. All of these references hint at and make use of aspects of reduction to a quotient space (especially notions such as "center of mass coordinates" and a V that depends only on relative configurations), but none of them fully explore it. The focus of those papers is generally more on the network structure and developing useful design and analysis rules based on the

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graph structure associated with the network.

A very closely related reference is [6] where the formation control problem for a group of robots is abstracted in a manner where the control of a team of robots with kinematic dynamics is formulated in terms of a low-dimensional trivial fiber bundle type structure composed of a group component and shape component for the formation. Then individual robot control laws are designed for convergence to the desired shape. This work contrasts to that in that we do not restrict the dynamics to be first order and we do not abstract out the relative position of each robot to its neighbors. Simply put, we "do reduction" on the formation control problem.

A second closely-related set of publications and body of work is due to Leonard [7], [8], involving the use of potential function for formation control and, especially in the case of [8] making use of classical reduction theory from mechanics for coordinated control of rigid bodies. In contrast, this work does not make use of the the properties of the system as a mechanical (or specifically Hamiltonian) system, but rather allows for more generic systems with Lie group symmetries (in the spirit of [2], rather than the mechanics of the work of, for example, Marsden [9] among his many publications).

B. Discrete Symmetries

Some of our prior work considers symmetric control systems wherein the system is composed of repeated instances of identical components [10], [11], which is to some degree a nonlinear extension of [12], [13]. The main focus of [10] is to exploit the discrete symmetric structure of the system to formulate results for stability that are independent of the number of components in the system and robustness results which also naturally follow from the symmetric nature of the system. The main focus of [11] is to extend such results to approximately symmetric systems. The main utility of the stability results is one of scalability. If the system is stable for a given number of components, stability is then guaranteed for a larger system composed of the same type of components which are interconnected in a manner consistent with the smaller system.

The main goal of this current work is to bring together the benefits of both the continuous and discrete symmetry results.

II. SYMMETRIC SYSTEMS

Stability results based on both continuous and discrete symmetries are outlined in this section.

A. Discrete Symmetries

Consider the system illustrated in Figure 1 where each node in the graph represents an agent and each edge represents interactions among the agents. The v-signals represent inputs to an agent, the subscript denotes the agent to which the input goes and the superscript denotes the relative relationship of the agent from which the input comes, e.g., v_1^{-2} indicates an input to agent 1 from agent N (located two agents counter-clockwise from it). The w-signals indicate

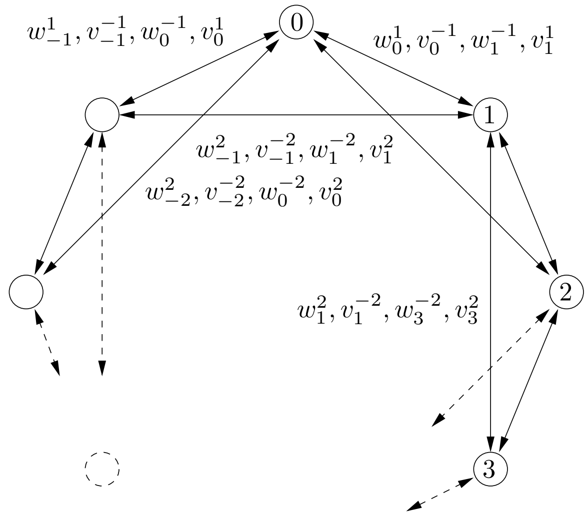

Fig. 1. System with discrete symmetric structure.

outputs from an agent, with similarly-defined sub- and superscripts, e.g., w_0^1 is the output from agent 0 directed to agent 1, etc.

As detailed in [10], if the inputs and outputs are related with sufficient regularity, then the system has *periodic interconnections*. If the dynamics of each agent are the same and the system has periodic interconnections, then the system is a *symmetric system* and an equivalence class of symmetric systems can be defined for different numbers of agents. Essentially the idea is, for example, in Figure 1, as long as the basic relationship between agents is defined and there is sufficient regularity to the structure, then all systems of the form illustrated are equivalent in some sense regardless of the value of N. If the system has a positive-definite Lyapunov function with a negative definite derivative and the Lyapunov function can be appropriately decomposed into a sum of terms corresponding to each agent, then it is possible to show that stability of only one system in the equivalence class implies stability for all members of the equivalence class.

As a specific example, consider the controlled dynamics of each agent to be

$$\ddot{x}_i = u_{x,i} = -\dot{x}_i - \sum_{j \in \mathcal{N}_i} (x_i - x_j) d_{ij}$$
$$\ddot{y}_i = u_{y,i} = -\dot{y}_i - \sum_{j \in \mathcal{N}_i} (y_i - y_j) d_{ij}$$

with

$$d_{ij} = (x_i - x_j)^2 + (y_i - y_j)^2 - \hat{d}_{ij},$$

where $\hat{d}_{ij}$ is the square of the desired distance between agents i and j, and $\mathcal{N}_i$ is the set of neighboring agents that agent i needs to be a designated distance away from to be in a desired formation. In the example in Figure 1, $\mathcal{N}_i$ contains the two agents in the counter-clockwise direction from i and the two agents in the clock-wise direction from i.

For this system, an obvious candidate Lyapunov function of

$$V_i = \frac{1}{2}(\dot{x}_i^2 + \dot{y}_i^2) + \frac{1}{8}\sum_{j\in\mathcal{N}_i} d_{ij}^2,$$

where the total Lyapunov function for a total of N agents is defined as $\sum_{i=1}^{N} V_i$. Then

$$\dot{V} = -\sum_{i=1}^{N}(\dot{x}_i^2 + \dot{y}_i^2).$$

This is negative semi-definite, which of course means that we cannot infer stability from Lyapunov's Theorem. One might infer stability-like properties from LaSalle's Principle because the largest invariant set for which $\dot{V} = 0$ is the set of desired formations. However, it is not straight-forward to define an invariant compact set containing all of the desired formations, as the initial conditions play a significant role in determining where the formation will be in space. In [10] this was addressed in the examples by adding a term to the control law attracting the formation to be attracted to the origin, which allowed for the identification of an invariant compact set which led to the appropriate use of LaSalle's Principle. Then, using the discrete symmetry, scaling of $\mathcal{O}(0)$ was obtained where the order is with respect to the number of agents in the system.

In the present efforts, we want to project the dynamics onto the quotient space defined by the SE(2) symmetry of the system to allow for similar results without the need for a term attracting the formation to the origin.

B. *Continuous Symmetry*

The goal of the present work is to make use of the SE(2) symmetry present in the problem to rigorously factor out the dependence of the dynamics and stability analysis on the explicit position of the agents and instead only depend on the relative position between them. To show some of the details of this approach, we consider a simple two-agent model using the same control approach as above.

For the two agent system, there is only one distance term and control laws simplify to

$$\begin{aligned}
\ddot{x}_1 &= -\dot{x}_1 - d_{12}(x_1 - x_2)\\
\ddot{y}_1 &= -\dot{y}_1 - d_{12}(y_1 - y_2)\\
\ddot{x}_2 &= -\dot{x}_2 - d_{21}(x_2 - x_1)\\
\ddot{y}_2 &= -\dot{y}_2 - d_{21}(y_2 - y_1)
\end{aligned}$$

with

$$d_{12} = (x_1 - x_2)^2 + (y_1 - y_2)^2 - \hat{d}_{12} = d_{21}.$$

Note that for two agents in a plane, the system on $\mathbb{R}^2 \times T\mathbb{R}^2 \times \mathbb{R}^2 \times T\mathbb{R}^2$ is parameterized by $x_1, x_2, \dot{x}_1, \dot{x}_2, y_1, y_2, \dot{y}_1, \dot{y}_2$. The system has a three-dimensional SE(2) symmetry corresponding to translation in the x- and y-directions and rotation. Therefore, there are a total of five quotient space variables and an easy way to define them is to consider terms obviously invariant with respect to the group actions

$$\begin{aligned}
g_x f(x,y) &= f(x+\epsilon, y)\\
g_y f(x,y) &= f(x, y+\epsilon)\\
g_\theta f(x,y) &= f(x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta).
\end{aligned}$$

It is easy to show that the distance, d_{12}, is invariant to the group actions. The derivative of the distance is also invariant to the group actions as is a third variable that is a function of the distance and its derivative. Finally, the fourth and fifth variables are quantities related to the overall linear and angular momentum of the system. Hence, we take

$$\begin{aligned}
q_1 &= d_{12} = (x_1 - x_2)^2 + (y_1 - y_2)^2 - \hat{d}_{12}\\
q_2 &= (x_1 - x_2)(\dot{x}_1 - \dot{x}_2) + (y_1 - y_2)(\dot{y}_1 - \dot{y}_2)\\
q_3 &= (\dot{x}_1 - \dot{x}_2)^2 + (\dot{y}_1 - \dot{y}_2)^2 + d_{12}^2\\
q_4 &= \dot{x}_1^2 + \dot{x}_2^2 + \dot{y}_1^2 + \dot{y}_2^2 + \frac{1}{2}d_{12}^2\\
q_5 &= (y_1 - y_2)(\dot{x}_1 + \dot{x}_2) - (x_1 - x_2)(\dot{y}_1 + \dot{y}_2),
\end{aligned}$$

and the dynamics are given by

$$\begin{aligned}
\dot{q_1} &= 2q_2\\
\dot{q_2} &= -2\hat{d}_{12}q_1 - q_2 + q_3 - 3q_1^2\\
\dot{q_3} &= -2q_3^2 + 2q_1^2\\
\dot{q_4} &= -2q_4^2 + q_1^2\\
\dot{q_5} &= -q_5 + \frac{q_2 q_5}{q_1 + \hat{d}_{12}} -
\end{aligned}$$

$$\frac{\sqrt{(q_2^2 - (q_3 - q_1^2)(q_1 + \hat{d}_{12}))(q_5^2 + (q_3 - 2q_4)(q_1 + \hat{d}_{12}))}}{q_1 + \hat{d}_{12}}.$$

In order to consider stability on the quotient space, consider the candidate Lyapunov function

$$V = \frac{1}{2}\begin{bmatrix} q_1 & q_2 \end{bmatrix}\begin{bmatrix} 2 & \frac{1}{4}\\ \frac{1}{4} & 2 \end{bmatrix}\begin{bmatrix} q_1\\ q_2 \end{bmatrix} + q_3^2 + \frac{1}{2}q_4^2 + \frac{1}{2}q_5^2,$$

which gives

$$\begin{aligned}
\dot{V} = &\begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix}\begin{bmatrix} -\frac{1}{8} & -\frac{1}{8} & \frac{1}{8}\\ -\frac{1}{8} & -1 & 1\\ \frac{1}{8} & 1 & -2 \end{bmatrix}\begin{bmatrix} q_1\\ q_2\\ q_3 \end{bmatrix}\\
&- \frac{3}{2}q_4^2 - \frac{3}{8}(q_1^2 + q_1)^2 - 3\left(\sqrt{6}q_1^2 + \frac{1}{\sqrt{6}}q_2\right)^2\\
&- 2(q_1^2 - q_3)^2 - \frac{1}{2}(q_1^2 - q_4)^2 + \frac{167}{8}q_1^4 - q_5^2 + \frac{q_5 q_2 q_5}{q_1 + \hat{d}_{12}} -
\end{aligned}$$

$$\frac{q5\sqrt{(q_2^2 - (q_3 - q_1^2)(q_1 + \hat{d}_{12}))(q_5^2 + (q_3 - 2q_4)(q_1 + \hat{d}_{12}))}}{q_1 + \hat{d}_{12}}.$$

Using

$$\begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix}\begin{bmatrix} -\frac{1}{8} & -\frac{1}{8} & \frac{1}{8}\\ -\frac{1}{8} & -1 & 1\\ \frac{1}{8} & 1 & -2 \end{bmatrix}\begin{bmatrix} q_1\\ q_2\\ q_3 \end{bmatrix} \leq \lambda_{max}(q_1^2 + q_2^2 + q_3^2)$$

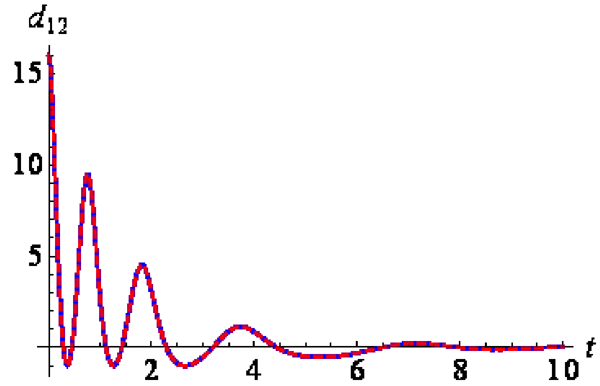

Fig. 2. Distance computation comparing full and reduced dynamics.

then (substituting the numerical values)

$$\begin{aligned}\dot{V} \leq &-0.11(q_1^2+q_2^2+q_3^2) - \frac{3}{2}q_4^2 - \frac{3}{8}(q_1^2+q_1)^2 \\ &-3\left(\sqrt{6}q_1^2 + \frac{1}{\sqrt{6}}q_2\right)^2 - 2(q_1^2-q_3)^2 - \frac{1}{2}(q_1^2-q_4)^2 \\ &+\frac{167}{8}q_1^4 - q_5^2 + \frac{q_5q_2q_5}{q_1+1} - \\ &\frac{q_5\sqrt{(q_2^2-(q_3-q_1^2)(q_1+1))(q_5^2+(q_3-2q_4)(q_1+1)))}}{q_1+1},\end{aligned}$$

when $\hat{d}_{12}=1$.

In order to guarantee that the expression is negative definite, $q_1^2 < 0.005135$. Therefore $q_2q_5^2/((q_1+\hat{d}_{12}) \leq 1.0772q_2q_5^2$ and

$$\begin{aligned}-q_5&\frac{\sqrt{(q_2^2-(q_3-q_1^2)(q_1+1))(q_5^2+(q_3-2q_4)(q_1+1))}}{q_1+1} \\ &\leq -0.933129q_5 \times \\ &\sqrt{(q_2^2-(q_3-q_1^2)(q_1+1))(q_5^2+(q_3-2q_4)(q_1+1))}\end{aligned}$$

Therefore

$$\begin{aligned}\dot{V} \leq &-0.1072(q_1^2+q_2^2+q_3^2) - \frac{3}{2}q_4^2 - \frac{3}{8}(q_1^2+q_1)^2 \\ &-3\left(\sqrt{6}q_1^2 + \frac{1}{\sqrt{6}}q_2\right)^2 - 2(q_1^2-q_3)^2 - \frac{1}{2}(q_1^2-q_4)^2 \\ &+\frac{167}{8}q_1^4 - q_5^2 + 1.0772q_2q_5^2 - 0.933129q_5 \times \\ &\sqrt{(q_2^2-(q_3-q_1^2)(q_1+1))(q_5^2+(q_3-2q_4)(q_1+1))},\end{aligned}$$

which is negative definite when $1.0772q_2 \leq \theta$, where θ is between 0 and 1, and when the square root term acts as a perturbation on the system, which occurs when the values are close enough to the origin.

An simulation illustrates both the stability of the dynamics and validity of the reduced dynamics. In Figure 2, the distance metric d_{12} is computed two different ways. The blue line corresponds to solving the system using the original (full) dynamics, while the dashed red line corresponds to the quotient space dynamics.

III. CONCLUSIONS

In this work we aim to combine reduction results for formation control problems characterized by both discrete and continuous symmetries. Reduced dynamics for continuous symmetries are beneficial for formation control beyond the reduction in dimension because of the simplifications in stability analysis which arise because a great many relative equilibria are reduced to equilibrium points. Furthermore, in engineering, many multi-agent systems are composed of identical agents, and hence making use of the discrete symmetry present in such problems is beneficial in terms of computational complexity and system and control design. This extended abstract presented our initial steps in combining the two approaches.

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