Towards General Results in Bifurcations in Optimal Solutions for Symmetric Distributed Robotic Formation Control

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***Abstract*— The modern trend of increased integration of formerly disparate systems is necessitating the development of advanced tools to study large-scale complex systems. This paper studies bifurcations in solution to an optimal control problem for robotic formation control. Robotic formation control is an excellent system-integration prototype problem because the scale of the problem can grow rapidly with increased numbers of robots, but the system retains some degree of heterogeneity which makes its study manageable. Our prior efforts have numerically studied the bifurcations for particular systems and performed an asymptotic analysis on those systems which provided insight into the rich and complicated structure of the solution space for such systems. The main contribution of this paper is an extension of the asymptotic analysis beyond the specific systems studied previously to a general class of systems. In both the specific and general cases, we show that as a system parameter is varied, the number of solutions increases from an unique solution to an infinite number of expected solutions, which bears resemblance to the cascade of period-doubling bifurcations typical of a dynamical system that exhibits chaos.**

I. INTRODUCTION

This paper studies the optimal trajectories for a formation of robots moving between specified initial and final formations. A combination of the control effort and the deviation from a desired formation is minimized in the solution. The optimal path is determined by solving a two-point boundary value problem for a set of second-order ordinary differential equations. We show that a bifurcation structure is present in the solutions as the relative emphasis between the control effort and maintaining the formation is varied. This structure is characterized by a unique solution when the control effort is more heavily weighted and an increasing number of solutions as the formation is more heavily weighted. Our prior work in [3], [4], [12] showed this for specific systems, and the main contribution of this paper is to extend those results to a more general class of systems. In [2] the initial steps toward generalizing the results were taken, but this paper further extends those results to more general cost functions for the inputs in the optimization.

The existence of multiple nontrivial solutions of boundary value problems for nonlinear second order ordinary differential equations has been investigated by others, but unfortunately none of those results are directly applicable to our problem. For example, for

$$\ddot{x} + a(t) f(x) = 0, \quad x(0) = 0, \quad x(1) = 0,$$

the properties of the solutions depend on the limiting behavior of the function $f(x)$. The existence of *positive* solutions of the equation with linear boundary conditions was studied in [6]. They showed that for

$$f_0 = \lim_{s \to +0} \frac{f(s)}{s}, \qquad f_\infty = \lim_{s \to +\infty} \frac{f(s)}{s},$$

the existence of at least one positive solution in the case of either *superlinearity* ($f_0 = 0, f_\infty = \infty$) or *sublinearity* ($f_0 = \infty, f_\infty = 0$). The results in [5], showed that there are at least two positive solutions with superlinearity at one end (zero or infinity) and sublinearity at the other end. In [11] and [9] conditions based on the ratio $f(s)/s$ for the existence (or nonexistence) of solutions were determined. The main result indicates that there are at least k solutions if the ratio $f(s)/s$ crosses the k eigenvalues of the associated eigenvalue problem. The existence, nonexistence, and multiplicity of positive solutions of the boundary value problem is determined using the fixed-point theorem of cone expansion/compression type in [10].

The rest of this paper is organized as follows. Section II presents the system under consideration as well as numerically-determined bifurcation results. Section III presents an asymptotic analysis which provides a theoretical for the observed structure of the bifurcation diagram for this system. Section IV presents an extension of this analysis from the specific case considered to a broad, general class of systems. Finally, Section V presents conclusions and future work.

II. ROBOTIC FORMATION SYSTEM DYNAMICS

This section first presents the system under consideration and then presents the results of a numerical investigation.

A. Robot Fleet System

We consider a canonical fully actuated two-degree of freedom system

$$\dot{x} = u_1, \quad \dot{y} = u_2. \tag{1}$$

which is a standard model for simple rolling-type robotic systems. We consider a group of n of robots with states (x_i, y_i), $i \in \{1, \ldots, n\}$ and try to find the control inputs $u_{i_1}(t), u_{i_2}(t)$ for each robot i, which steer a formation start formation to a goal formation, while maintaining, to some

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degree, the formation throughout the execution of the motion. This is achieved by minimizing the functional

$$J = \int_0^{t_f} \underbrace{\sum_{i=1}^{n} \left((u_{i_1})^2 + (u_{i_2})^2 \right)}_{\text{minimize control effort}} + k \underbrace{\sum_{i=1}^{n-1} \left(d_i - \overline{d} \right)^2}_{\text{maintain formation}} \, dt$$

subject to the dynamic constraints in Equation 1, where n is the number of robots, $d_i = \sqrt{(x_i - x_{i+1})^2 + (y_i - y_{i+1})^2}$ is the distance between the ith and $(i+1)$th robots, $\overline{d}$ is the desired distance between two adjacent robots, and k is a weighting factor. When k is small the control effort is more important and when it is large maintaining the formation is more important.

Standard methods such as calculus of variations or Pontryagin's maximum principle give $u_{i_1} = p_{i_1}/2$, $u_{i_2} = p_{i_2}/2$, and equations of motion[1]

$$\begin{aligned}
\dot{x}_i &= \frac{1}{2} p_{i_1} \\
\dot{y}_i &= \frac{1}{2} p_{i_2} \\
\dot{p}_{i_1} &= \frac{2k \left(x_i - x_{i-1}\right) \left(d_{i-1} - \overline{d}\right)}{d_{i-1}} + \frac{2k \left(x_i - x_{i+1}\right) \left(d_i - \overline{d}\right)}{d_i} \\
\dot{p}_{i_2} &= \frac{2k \left(y_i - y_{i-1}\right) \left(d_{i-1} - \overline{d}\right)}{d_{i-1}} + \frac{2k \left(y_i - y_{i+1}\right) \left(d_i - \overline{d}\right)}{d_i}.
\end{aligned} \tag{2}$$

For space limitations we focus on the specific boundary conditions

$$\begin{aligned}
x_i(0) &= c + (i-1)\overline{d}, \quad & y_i(0) &= 0 \\
x_i(1) &= 0, & y_i(1) &= c + (i-1)\overline{d},
\end{aligned} \tag{3}$$

where c is a constant. Hence the initial formation has the robots evenly spaced along the x-axis and the final formation has the robots evenly spaced along the y-axis as is illustrated in any of the following simulation illustrations.

Remark 1: Observe that straight-line trajectories connecting the initial and final formation locations of each robot will not maintain the desired distance between the robots. Therefore, solutions that minimizes the control effort, which are straight lines, will not maintain the desired formation. Hence relative weighting of the two terms in the cost function will be important and affect the solution, and, as the rest of this paper shows, is a means to characterize the nature and number of solutions to this problem. $\diamond$

B. Simulation Results

Some of our prior publications have presented extensive numerical investigations using various numerical solution methods for the boundary value problem[2] for this and other systems for various numbers of robots in the group [3], [4], [12]. Here we present the bifurcation results and the following sections provide a theoretical explanation for the results and extends that justification to a very broad class of systems.

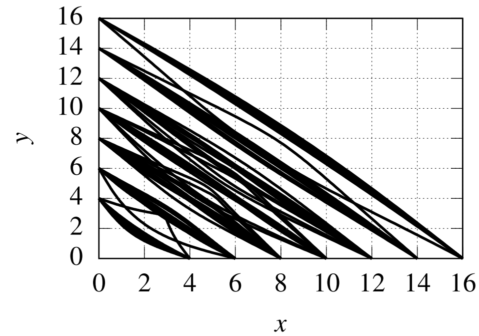

Fig. 1. Eleven solutions for a seven-robot system with $k = 23$.

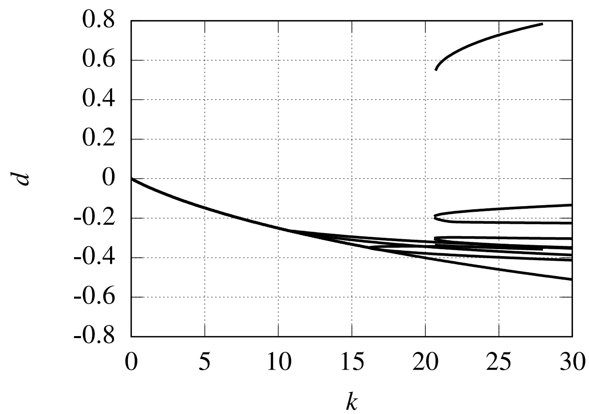

Fig. 2. Bifurcation diagram for robot number one in a seven-robot system.

Because the problem in Equations 2 and 3 is a nonlinear two-point boundary value problem, unique solutions do not necessarily exist. In fact, many solutions may exist, depending on the value of k, the parameter that is the relative importance of maintaining the formation *versus* the control effort. For example, for seven robots and $k = 23$ there are at least 11 different solutions, which are all illustrated in Figure 1.

To illustrate the geometric structure of the relationship among the solutions, we compare each solution to the $k = 0$ case. When $k = 0$ the solutions are straight lines that connect the starting and ending points. The solutions are straight because $k = 0$ corresponds to only considering the control effort and not the formation, and hence the robot will go in the shortest possible path connecting the starting and ending locations. While it is not necessarily a perfect measure, if we measure the distance above or below the $k = 0$ solution at one-quarter of the time between the start and end, and plot that measure for various k values, we can illustrate some of the geometry of the solution space. Figures 2 through 8 illustrate that for a seven-robot system. The d value is the of the solution from the $k = 0$ solution. Positive values of d are solutions above and negative values are below.

Remark 2: The main feature to note from these bifurca-

[1]Note: because they are the robots at the two ends, the last two equations in Equation 2 only have the second term when $i = 1$ and they only have the first term when $i = n$. This is because there is no 0 robot or $n+1$ robot.

[2]Specifically we have used the shooting method as well as finite difference methods.

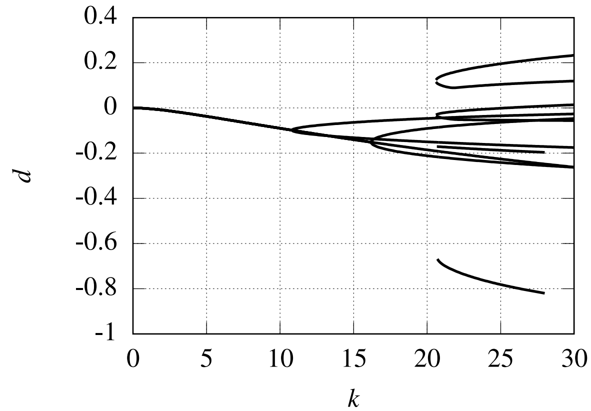

Fig. 3. Bifurcation diagram for robot number two in a seven-robot system.

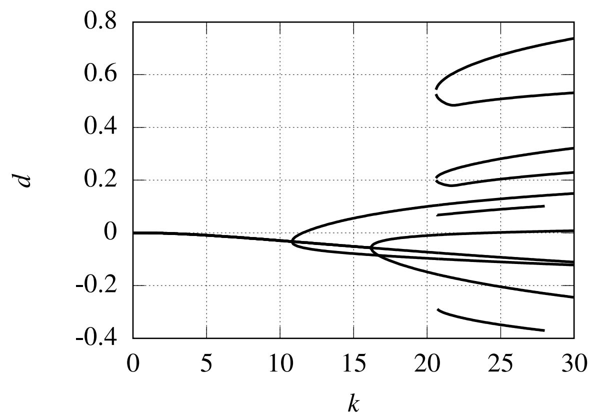

Fig. 4. Bifurcation diagram for robot number three in a seven-robot system.

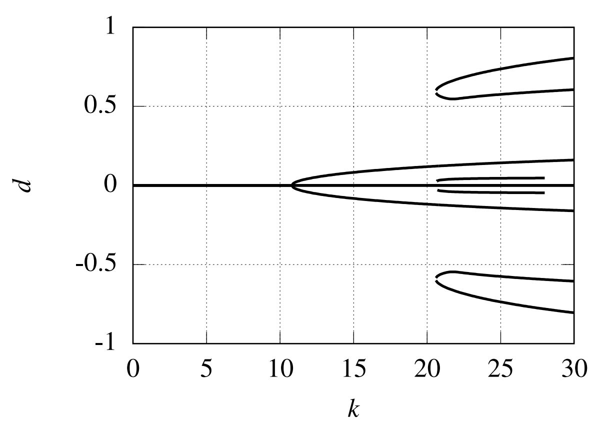

Fig. 5. Bifurcation diagram for robot number four in a seven-robot system.

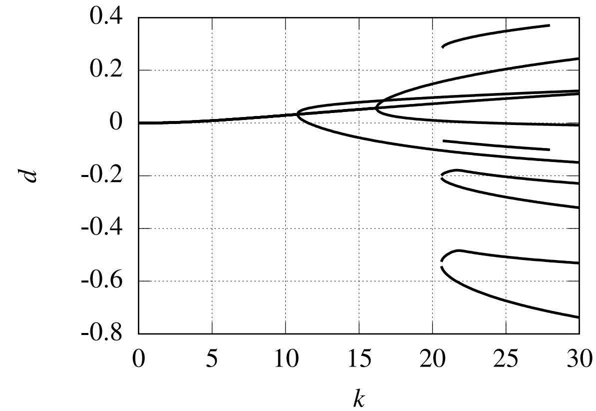

Fig. 6. Bifurcation diagram for robot number five in a seven-robot system.

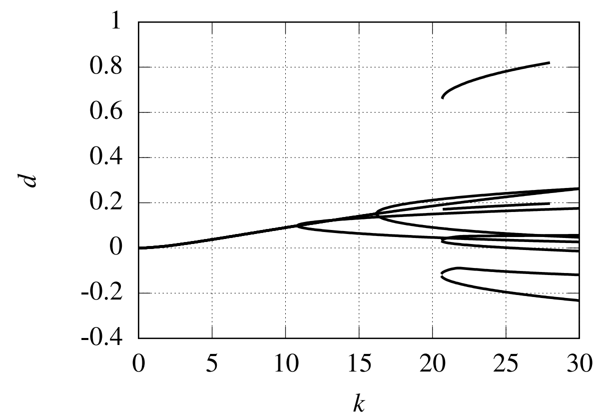

Fig. 7. Bifurcation diagram for robot number six in a seven-robot system.

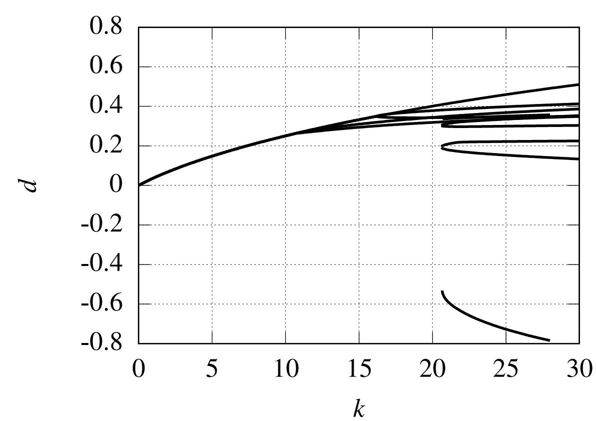

Fig. 8. Bifurcation diagram for robot number seven in a seven-robot system.

tion diagrams is the increasing number of solutions as k is increased. For a small k value, there is only one solution, and as k is increased, the number of solutions increases. This is similar to period-doubling bifurcations in scalar dynamical system transitioning to chaos. However, in the present case it is fundamentally very different because we are solving *boundary value problems* is contrast to ordinary differential equations. ◇

The remainder of this paper shows that as k is increased, an increasing number of solutions is expected. This has important ramifications for large scale robotic control because a proliferation of locally optimal solutions will make it difficult to apply optimization methods to integrated systems of many interacting robots.

III. ASYMPTOTIC ANALYSIS

Because the governing equations are nonlinear, global results are difficult to obtain and the bifurcation diagrams we present here and in the other references were all obtained via extensive numerical searches. Such a method does not preclude the existence of solutions we did not find.

This section presents a theoretical analysis that validates the qualitative dependence of the number of solutions on the bifurcation parameter. An asymptotic expansion is used to investigate the cases of very small k and very large k, with the focus of the analysis on the number of solutions. This analysis is consistent with the existence of a unique solution for small values of k and many solutions for very large k, which is the pattern indicated in the numerical results.

A. Emphasis on Control Effort: Small k

We use standard perturbation methods [7], [8] to determine a series expansion for solutions to Equations 2 for $k \ll 1$. Let

$$\begin{aligned}
x_i &= x_{i,0} + kx_{i,1} + k^2 x_{i,2} + k^3 x_{i,3} + \cdots + k^j x_{i,j} + \cdots,\\
y_i &= y_{i,0} + ky_{i,1} + k^2 y_{i,2} + k^3 y_{i,3} + \cdots + k^j y_{i,j} + \cdots,\\
p_{i_1} &= p_{i_1,0} + kp_{i_1,1} + k^2 p_{i_1,2} + k^3 p_{i_1,3} + \cdots + k_j p_{i_1,j} + \cdots,\\
p_{i_2} &= p_{i_2,0} + kp_{i_2,1} + k^2 p_{i_2,2} + k^3 p_{i_2,3} + \cdots + k_j p_{i_2,j} + \cdots,
\end{aligned}$$

and substitute into Equation 2, which results in a set of *linear* differential equations for each power of k. Space limitations prevent inclusion here, but the nature and structure of the asymptotic expansion for all orders of this system appears in [1].

The equation corresponding to k^0 terms gives the set of linear equations for the ith robot as

$$\begin{aligned}
\dot{x}_{i,0} &= \frac{1}{2} p_{i_1,0},\\
\dot{y}_{i,0} &= \frac{1}{2} p_{i_2,0},\\
\dot{p}_{i_1,0} &= 0,\\
\dot{p}_{i_2,0} &= 0,
\end{aligned}$$

with boundary conditions

$$\begin{aligned}
x_{i,0}(0) &= x_{1,0}(0) + (i-1)\overline{d}\\
y_{i,0}(0) &= 0\\
x_{i,0}(1) &= 0\\
y_{i,0}(1) &= y_{1,0}(1) + (i-1)\overline{d}.
\end{aligned}$$

These can be solved by direct integration with solutions

$$\begin{aligned}
x_{i,0} &= -x_{i,0}(0)t + x_{i,0}(0),\\
y_{i,0} &= y_{i,0}(1)t,\\
p_{i_1,0} &= -2x_{i,0}(0),\\
p_{i_2,0} &= 2y_{i,0}(1).
\end{aligned}$$

Remark 3: These are straight line, constant velocity solutions, which is expected because the 0th order equation will not contain k. Because k multiplies the formation term, this corresponds to only considering the control effort and giving no weight at all to the formation. Furthermore, the solutions are *unique*, validating the qualitative nature of the bifurcation diagrams presented above. ◇

The k^1th order equations are of the form

$$\begin{aligned}
\dot{x}_{i,1} &= \frac{p_{i_1,1}}{2}\\
\dot{y}_{i,1} &= \frac{p_{i_2,1}}{2}\\
\dot{p}_{i_1,1} &= 2\left(\frac{(x_{i,0}-x_{i-1,0})\,(d_{i-1,0}-\overline{d})}{d_{i-1,0}} + \frac{(x_{i,0}-x_{i+1,0})\,(d_{i,0}-\overline{d})}{d_{i,0}}\right)\\
\dot{p}_{i_2,1} &= 2\left(\frac{(y_{i,0}-y_{i-1,0})\,(d_{i-1,0}-\overline{d})}{d_{i-1,0}} + \frac{(y_{i,0}-y_{i+1,0})\,(d_{i,0}-\overline{d})}{d_{i,0}}\right),
\end{aligned} \tag{4}$$

where $d_{i,0}$ is the distance term as a function of the zeroth-order solutions.

The equations corresponding to higher orders of k are obtained similarly, but naturally grow in complexity and are omitted due to space limitations. As is typical for an asymptotic analysis, the differential equations for k^1 depend only on $x_{i,1}, y_{i,1}, p_{i_1,1}$ and $p_{i_2,1}$ as well as the lower order solutions, *but not the higher-order solutions*. Thus, the zeroth-order solutions appear as nonhomogenous terms in the first-order equations, and recursively so for the higher-order equations.

The solutions to the k^1 order equations are also unique. This is because the first-order costate equations only depend on the lower-order solutions they may be solved by direct integration. Once the first-order costates are determined, $x_{i,1}$ and $y_{i,1}$ may be obtained by direct integration and the four boundary conditions will can be satisfied by the four constants of integration. Because the 0th-order solutions are continuous and bounded, a unique solution for each integral exits.

Therefore, this is consistent with the solution in the neighborhood of the straight-line $k = 0$ zeroth-order solution being unique, which is furthermore consistent with the physical interpretation that when the formation weighting is zero, the only optimal solution is the one that minimizes the path length, which is a straight line.

Because the zeroth-order solutions are straight lines, only the first and nth equations will have non-zero solutions for the first-order equations because straight lines are such that effect of its neighbors on the ith robot cancel. This can be seen in Equation 4 because the right-hand side of the costate equations depend only on the 0th-order solutions. Thus the costate equations will have constant solutions and because the end robots do not have a neighbor on each side, they are the only robots that will have a first-order effect from k.

Space limitations prevent including the higher-order equations (an interested reader is referred to [1]), but the second from the end robots have a zero first-order solution and non-zero second-order solution, *etc*. Hence, the deviation from the straight-line solution is of increasingly higher order in k toward the middle of the formation. This is consistent with an intuitive idea that robots near the outside of the formation have greater flexibility in their path to move away from the straight line, and, in contrast, robots in the middle are "squeezed" by the formation.

B. Emphasis on Maintaining Formation: Large k

When k is very large, maintaining the distance between the robots is weighted more heavily in the cost function than the control effort and $\varepsilon = 1/k$ is used as the expansion parameter. As before, let

$$\begin{aligned} x_i &= x_{i,0} + \varepsilon x_{i,1} + \varepsilon^2 x_{i,2} + \varepsilon^3 x_{i,3} + \cdots + \varepsilon^j x_{i,j} + \cdots, \\ y_i &= y_{i,0} + \varepsilon y_{i,1} + \varepsilon^2 y_{i,2} + \varepsilon^3 y_{i,3} + \cdots + \varepsilon^j y_{i,j} + \cdots, \\ p_{i_1} &= p_{i_1,0} + \varepsilon p_{i_1,1} + \varepsilon^2 p_{i_1,2} + \varepsilon^3 p_{i_1,3} + \cdots + \varepsilon_j p_{i_1,j} + \cdots, \\ p_{i_2} &= p_{i_2,0} + \varepsilon p_{i_2,1} + \varepsilon^2 p_{i_2,2} + \varepsilon^3 p_{i_2,3} + \cdots + \varepsilon_j p_{i_2,j} + \cdots, \end{aligned}$$

and the resulting equations corresponding to ε^0 are

$$\begin{aligned} \dot{x}_{i,0} &= \frac{1}{2} p_{i_1,0} \\ \dot{y}_{i,0} &= \frac{1}{2} p_{i_2,0} \\ 0 &= \frac{2k\left(x_{i,0} - x_{i-1,0}\right)\left(d_{i-1,0} - \overline{d}\right)}{d_{i-1,0}} + \frac{2k\left(x_{i,0} - x_{i+1,0}\right)\left(d_{i,0} - \overline{d}\right)}{d_{i,0}} \\ 0 &= \frac{2k\left(y_{i,0} - y_{i-1,0}\right)\left(d_{i-1,0} - \overline{d}\right)}{d_{i-1,0}} + \frac{2k\left(y_{i,0} - y_{i+1,0}\right)\left(d_{i,0} - \overline{d}\right)}{d_{i0}}. \end{aligned}$$

The last two equations are equivalent to

$$(x_{i,0} - x_{i-1,0})^2 + (y_{i,0} - y_{i-1,0})^2 = \overline{d}^2, \qquad (5)$$

so that the limit for large k requires that the distance constraint be exactly maintained.

Since the third and fourth equations are algebraic, then the costates, p are unconstrained and therefore any path that maintains the desired distance between the robots and satisfies the boundary conditions is a solution to these equations. This makes intuitive sense because in the limit as $k \to \infty$, the control effort becomes negligible relative to the formation constraint. Therefore, as k becomes very large, the asymptotic analysis indicates that there is an *infinite* number of solutions. As long as the separation distance is maintained and the boundary conditions are satisfied, any path is optimal.

Remark 4: This analysis validates the right-hand extremes of the bifurcation diagrams illustrated in Figures 2 through 8 where an increasing number of possible solutions are obtained as k is increased. ◇

IV. EXTENSION OF RESULTS TO THE GENERAL CASE

Up to this point, the results in this section are for the specific example in this paper. The same qualitative nature of the number of solutions (a unique solution for small k and an increasing number of solutions as k is increased), has been observed in other systems as well [4], [12], which naturally leads to inquire as to whether this is a generic feature in such problems.

The main contribution of this paper is to show that this general feature will be present in a very broad class of problems. To do this, consider a much more general cost functional, which will naturally lead to a more general class of differential equations governing the dynamics of the system. It makes compelling engineering sense to minimize the control effort, so we generalize the formation function component of the functional.

Specifically, let

$$\begin{aligned} J = \int_0^{t_f} & g\left(u_{1_1}, u_{1_2}, \ldots, u_{n_1}, u_{n_2}\right) \\ & + k f(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n) dt \end{aligned}$$

where the function, f, is a differentiable function of the relative configuration of the robots that is minimized when the robots are in the desired formation. Calculus of variations yields $p_{i_1} = \partial g / \partial u_{i_1}$ and $p_{i_2} = \partial g / \partial u_{i_2}$ and

$$\begin{aligned} \dot{x}_i &= \left(\frac{\partial g}{\partial u_{i_1}}\right)^{-1} (p_{i_1}) \\ \dot{y}_i &= \left(\frac{\partial g}{\partial u_{i_2}}\right)^{-1} (p_{i_2}) \\ \dot{p}_{i_1} &= 2k \frac{\partial f}{\partial x_i} \\ \dot{p}_{i_2} &= 2k \frac{\partial f}{\partial y_i}. \end{aligned}$$

Following the same asymptotic analysis as above, it is clear that for $k \ll 1$, we have

$$\begin{aligned} \dot{x}_i &= \left(\frac{\partial g}{\partial u_{i_1}}\right)^{-1} (p_{i_1}) \\ \dot{y}_i &= \left(\frac{\partial g}{\partial u_{i_2}}\right)^{-1} (p_{i_2}) \\ \dot{p}_{i_1} &= 0 \\ \dot{p}_{i_2} &= 0 \end{aligned}$$

which will be satisfied if $p_{i_1} = \partial g / \partial u_{i_1}$ and $p_{i_2} = \partial g / \partial u_{i_2}$ satisfy the inverse function theorem and there exist constant costate values such that the boundary conditions are satisfied. This will usually be true in common engineering scenarios with sufficient regularity on the function g, for example positive functions with nonzero derivatives away from zero, such as sums of even powers of the inputs.

Also, when $k \gg 1$ we have

$$\dot{x}_i = \left(\frac{\partial g}{\partial u_{i_1}}\right)^{-1} (p_{i_1})$$
$$0 = 2k\frac{\partial f}{\partial x_i}$$
$$0 = 2k\frac{\partial f}{\partial y_i}$$

which is satisfied by *any* trajectory when the formation function is minimized. Hence, as was the case for the very specific system we considered, the same results hold for *any* formation function which is differentiable and minimized when the formation is satisfied. Specifically, there is a unique solution for small k and infinitely many solutions in the limit as $k \to \infty$, suggesting a cascade of bifurcations qualitatively the same as we found for the specific system considered in this paper.

V. CONCLUSIONS AND FUTURE WORK

This paper analyzed the nature and structure of multiple solutions to an optimal control problem for formation control for multiple mobile robots. Our previous results were limited to numerical investigations of specific systems, and the main contribution of this paper is a theoretical component that indicates a relatively broad generality to the results. The subject matter of this paper is important in both robotics and the broader system integration communities because various forms of optimization are common in both areas and an understanding of the global structure of the solution space will lead to more efficient and optimal solution methods.

When considering the trade-off between minimizing control effort and maintaining the desired formation, a complicated and rich solution bifurcation structure is present. By considering a specific system, numerical simulation results show a unique solution when a much greater weight was given to the control effort compared to the formation cost and when the formation is given a much greater weight, many solutions are present.

The main contribution of this paper was to extend the theoretical reach of these results beyond specific examples, which was a limitation of our prior work. Specifically, we showed that for a very general class of systems, if an optimization method is used for trajectory generation for fleets of robots, if the cost function contains “control effort” terms and “formation maintaining” terms, as increasing emphasis is placed on the formation terms, an increasing number of solutions is expected. Specifically, in the limit, *any* trajectory that maintains the formation will satisfy the optimization criterion.

There are several valuable avenues for future research.

- First, a systematic means for characterizing bifurcations for boundary value problems is needed. This seems to be an open problem in the mathematical research community without an obvious generalization for bifurcations of fixed points of dynamical systems because the entire solution, as opposed to an isolated equilibrium, is what bifurcates. This fundamentally arises because of the two-point boundary value problem nature of the optimization.
- Second, numerically efficient methods are necessary for searching for solutions. The shooting method and finite difference methods are only somewhat efficient for finding isolated solutions because they are iterative. Extending the methods to globally search for solutions is obviously problematic. Some initial results exist based on polynomial homotopy methods [4] but due to the manner in which the number of roots of polynomial systems grow with the order of the polynomial and dimension of the problem, such an approach does not seem to scale well.
- Finally, due to the utility in practical implementations, connecting these results with receding-horizon methods would be useful.

VI. ACKNOWLEDGMENTS

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