Fractional-Order Approximations to Implicitly-Defined Operators for Modeling and Control of Networked Mechanical Systems

Bill Goodwine

***Abstract*— This paper considers interconnected mechanical systems intended to be simplified prototypical Cyber-Physical Systems. In many modern engineered systems, there are many interacting components with coupled dynamics. Some of our prior work has shown that such systems exhibit fractional-order dynamics, which may serve as a concise model for such very high-order systems. This paper extends that prior work to a more general class of systems in which the operator describing the dynamics of the system can only be determined implicitly. A system-identification procedure indicates that these systems are also predominantly fractional-order as well, and hence it is likely that many large-scale CPS may be best modeled by fractional-order differential equations. In such cases, then, given the central role of system models in control theory, development and awareness of such fractional-order dynamics in CPS are essential for controls engineers.**

I. INTRODUCTION AND MOTIVATION

Consider the system illustrated in Figure 1, which represents a tree of masses, springs and dampers, or can be considered a type of formation of robots wherein each robot controls its position relative to its neighbors in accordance with a potential or viscous-like relationship, as the case may be. In a prior publication [1] we showed that in the limit of an infinite number of generations the transfer function relating the position of the last generation to the position of the first generation is governed by a 1/2-order differential relationship and furthermore, that 1/2-order system is accurate for a finite and relatively small number of generation.

In this paper we consider what initially seems to be a simple modification of this system, but in fact, results in the surprising instance that the operator representing the force relationship between the first and last generations is only implicitly defined [2]. The main result in this paper is that this implicitly-defined operator seems to be also best approximated by a fractional-order system. Specifically, consider the system in Figure 2, where at each generation instead of there being one spring and one damper, there are two springs and one damper. This paper will investigate the nature of the dynamics as the number of springs and/or dampers in each generation is changed. Interestingly, the case of one spring and one damper, which generated 1/2-order dynamics, is the only one that can apparently be expressed explicitly in operator form. Hence, the nature of different types of approximations for such systems arises, and is the contribution of this paper.

Now we determine the nature of the operator that relates the relative displacement of the first and last generations to the force generated by the network between the generations. In the limit as the number of generations goes to infinity,

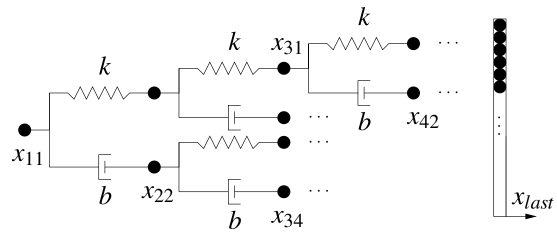

Fig. 1. Structure of 1/2-order system.

the system is characterized by a *self-similarity*. Specifically, with an infinite number of generations, *the transfer function from* x_{11} *to* x_{last} *is equal to the transfer function from any other* x_{ij} *to* x_{last}.

If we let

$$G_1(s) = \frac{1}{k} \qquad G_2(s) = \frac{1}{bs}$$

and using usual parallel and series rules for interconnected mechanical components and denote the transfer function from any node to the last node by $G_\infty(s)$, the self-similarity in the system is represented by

$$G_\infty(s) = \frac{X_{11}(s) - X_{last}(s)}{F(s)} = \frac{1}{\frac{1}{G_1(s)+G_\infty(s)} + \frac{1}{G_2(s)+G_\infty(s)}},$$

where $F(s)$ is the force needed to displace the two ends of the network by $X_{11}(s) - X_{last}(s)$. Solving for $G_\infty(s)$ explicitly (because the actual algebraic manipulations will matter subsequently):

$$\begin{aligned} G_\infty(s) &= \frac{1}{\frac{(G_2(s)+G_\infty(s))+(G_1(s)+G_\infty(s))}{(G_1(s)+G_\infty(s))(G_2(s)+G_\infty(s))}} \\ &= \frac{G_\infty^2(s) + (G_1(s)+G_2(s))\,G_\infty(s) + G_1(s)G_2(s)}{G_1(s)+G_2(s)+2G_\infty(s)} \end{aligned}$$

which clearing the denominator and moving all the terms to one side simplifies to

$$G_\infty^2(s) - G_1(s)G_2(s) = 0$$

or

$$G_\infty(s) = \sqrt{G_1(s)G_2(s)} = \frac{1}{\sqrt{kbs}}. \qquad (1)$$

The square root of s indicates 1/2-order dynamics and our work in [1] validates this. Only the positive solution is kept if the spring and damper constants represent physical components where they must be positive. In the next section which reviews fractional calculus, we show that the operator $s^{1/2}$ can be explicitly represented in the time domain.

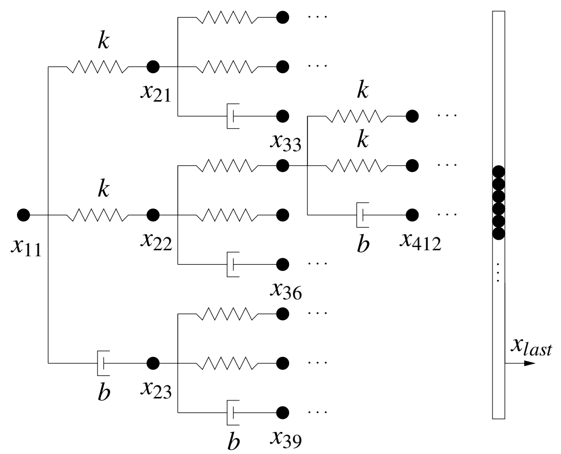

Fig. 2. Structure of modified system.

In contrast, now consider the system in Figure 2. In this case the parallel and series composition gives

$$G_\infty(s) = \frac{X_{11}(s) - X_{last}(s)}{F(s)} = \frac{1}{\frac{2}{G_1(s)+G_\infty(s)} + \frac{1}{G_2(s)+G_\infty(s)}}.$$

In this case, simplifying leads to

$$2G_\infty^2(s) + G_2(s)G_\infty(s) - G_1(s)G_2(s) = 0. \tag{2}$$

This is quadratic in the unknown, $G_\infty(s)$ with known operator coefficients, $G_1(s)$ and $G_2(s)$. We can formally use the quadratic equation to obtain an expression for $G_\infty(s)$, but it can not be expressed in the time domain by a fractional time derivative. As such, we must resort to approximations.

The usual approach in controls would be a Padé approximation or something similar for nonrational transfer functions. However, in this paper we will show that combinations of fractional operators appear to provide a good alternative representation warranting further investigation.

II. FRACTIONAL-ORDER CALCULUS AND FRACTIONAL-ORDER SELF-SIMILAR SYSTEMS

The system in Figure 1, which was inspired by a viscoelastic model from [3], [4], was accurately modeled in [1] by a fractional-order differential equation with a term of order $1/2$. Concise models with few terms are advantageous in controls and robotics, and thus fractional-order dynamics can be exploited for efficient and concise models of high-order systems such as interacting, multi-agent networks.

Research in control of multi-robot systems has many problems of great interest (see, for example, [5]–[9]). Some of the prior work by the author considers exact model reduction for systems with symmetries [10]–[13]. Fractional calculus dates back to near the beginning of calculus. Books on the mathematical fundamentals and engineering applications include [14]–[16], and review articles include [17], [18]. One study along a similar line of inquiry to that in this work is [19], [20], concerning formation control of fractional systems. In those references, however, the individual components within the system are fractional. In contrast in this paper, the fractional dynamics arise from the structure of the agents' interactions. Other papers from the authors on fractional calculus in engineering are [21], [22].

Given a function $f(t)$ with first derivative $f^{(1)}(t)$ and second derivative $f^{(2)}(t)$, it is natural to ask whether there are operators "in between" these such as

$$f^{(1/2)}(t) = \frac{d^{1/2}f}{dt^{1/2}}(t)$$

that generalize the concept of a derivative beyond the typical integer orders.

While there are closed-form solution techniques for fractional-order differential equations, as overviewed in the authors' prior works and the literature, numerical approximations are often necessary. Consider the following definitions of the first and second derivatives of a function

$$\frac{df}{dt}(t) = \lim_{\Delta t \to 0} \frac{f(t) - f(t-\Delta t)}{\Delta t}$$
$$\frac{d^2f}{dt^2}(t) = \lim_{\Delta t \to 0} \frac{f(t) - 2f(t-\Delta t) + f(t-2\Delta t)}{(\Delta t)^2},$$

or for an integer value of n,

$$\frac{d^n f}{dt^n}(t) = \lim_{\Delta t \to 0} \frac{\sum_{0 \le m \le n} (-1)^m \binom{n}{m} f\left(t + (n-m)\Delta t\right)}{(\Delta t)^n},$$

with the binomial coefficient given by

$$\binom{n}{m} = \frac{n!}{m!\,(n-m)!}.$$

Recall that the gamma function is a generalization of the factoria, and hence we can perform the substituiton

$$\binom{\alpha}{m} = \frac{\Gamma(\alpha+1)}{\Gamma(m+1)\Gamma(\alpha-m+1)}, \tag{3}$$

which yields the *Grünwald-Letnikov derivative*:

$$\frac{d^\alpha f}{dt^\alpha}(t) = \lim_{\Delta t \to 0} \frac{1}{(\Delta t)^\alpha} \sum_{j=0}^{\infty} (-1)^j \binom{\alpha}{j} f\left(t + (\alpha - j)\Delta t\right). \tag{4}$$

If $\Delta t \ll 1$ and $t = m\Delta t$, then the quantity $\alpha\Delta t$ in the argument of f can be neglected, and, as is typical in controls, assuming initial conditions of zero, the approximation becomes

$$\frac{d^\alpha f}{dt^\alpha}(t) \approx \frac{1}{(\Delta t)^\alpha} \sum_{j=0}^{m} (-1)^j \binom{\alpha}{j} f\left(t - j\Delta t\right), \tag{5}$$

which is an efficient way to formulate solutions of fractional-order differential equations.

Remark 1: An important contrast between fractional- and integer-order derivatives is that fractional-order derivatives cannot be computed from local information only. The summation in Equation 4 is evidence that all past values of a function are part of the computation of its fractional derivative

Referring to the system in Figure 1 with the transfer fuction in Equation 1, if we take the position of mass x_{11} as the input and the location of the last generation, x_{last}

as the output, then we have the equivalent time-domain representation of

$$m_{last}\frac{d^2}{dt^2}x_{last}(t) = \sqrt{kb}\frac{d^{1/2}}{dt^{1/2}}\left(x_{11}(t) - x_{last}(t)\right)$$

or

$$m_{last}\frac{d^2 x_{last}}{dt^2}(t) + \sqrt{kb}\frac{d^{1/2}x_{last}}{dt^{1/2}}(t) = \sqrt{kb}\frac{d^{1/2}x_{11}}{dt^{1/2}}(t).$$

This can be solved using the numerical approximation given by Equation 5 and our results in [1] showed that the fractional solution was a good approximation even for the system with a relatively small number of generations.

In this paper, considering systems like in Figure 2, however, we can not directly take this approach because we can not explicitly solve Equation 2 for a fractional derivative term. The rest of this paper shows that using a fractional-order system identification procedure, systems like the one in Figure 2 are still predominantly fractional, and we can use fractional-order models for them.

III. Frequency Response of Self-Similar Systems

In this section we construct frequency-response diagrams for self-similar systems like the one illustrated in Figure 2. We generalize the type of system we consider by considering a system where there are n springs and m dampers on each generation, where Figure 2 illustrates the specific case where $n = 2$ and $m = 1$. In this case we have

$$G_\infty(s) = \frac{X_{11}(s) - X_{last}(s)}{F(s)} = \frac{1}{\frac{n}{G_1(s)+G_\infty(s)} + \frac{m}{G_2(s)+G_\infty(s)}}.$$

In this case, simplifying leads to

$$(n+m-1)\,G_\infty^2(s) + \left[(m-1)\,G_1(s) + (n-1)\,G_2(s)\right]G_\infty(s) - G_1(s)G_2(s) = 0. \quad (6)$$

As before, $G_\infty(s)$ is the transfer function relating the relative displacement of the two ends of the network to the force required for the displacement. This equation for $G_\infty(s)$ has operator coefficients, which can be formally solved using the quadratic equation to give

$$G_\infty(s) = \frac{1}{2\,(n+m-1)}\Big[(1-m)\,G_1(s) + (1-n)\,G_2(s) \pm \Big(\left[(m-1)\,G_1(s) + (n-1)\,G_2(s)\right]^2 + 4\,(n+m-1)\,G_1(s)G_2(s)\Big)^{1/2}\Big]. \quad (7)$$

In the special case where $n = m = 1$, this reduces to the 1/2-order system. Figure 3 compares the frequency response for the case where $n = m = 1$ for a finite system with 50 generations (blue) with the self-similar, infinite-generation case (green). In all the cases in this section $k = b = 1$.

In the case from Figure 2 where $n = 2$ and $m = 1$, this reduces to

$$G_\infty(s) = \frac{1}{4}\left[-G_2(s) \pm \sqrt{G_2^2(s) + 8G_1(s)G_2(s)}\right]$$

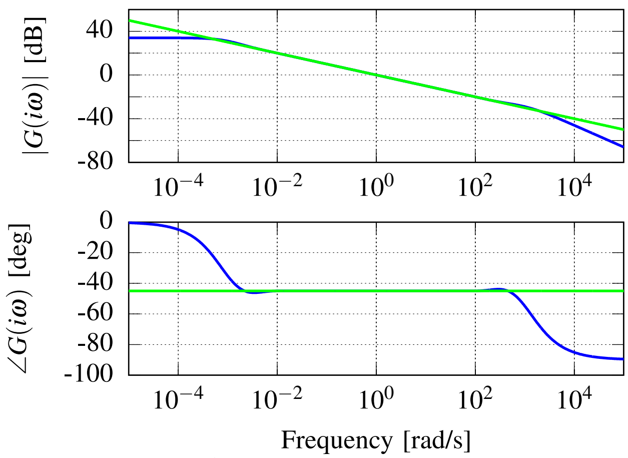

Fig. 3. Comparison of 50-generation system (blue) with self-similar system (green) when $n = m = 1$.

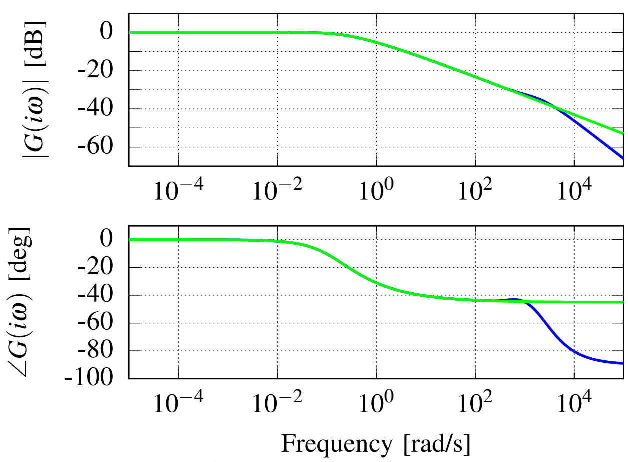

Fig. 4. Comparison of 50-generation system (blue) with self-similar system (green) when $n = 2$ and $m = 1$.

and in the specific case where $G_1(s) = 1/k$ and $G_2(s) = 1/(bs)$ this is

$$G_\infty(s) = \frac{-k \pm \sqrt{k^2 + 8bks}}{4bks}.$$

Figure 4 compares the frequency response for the case where $n = 2$ and $m = 1$ for a finite system with 50 generations (blue) with the self-similar, infinite-generation case (green). We emphasize that an important distinction between this case and the one-spring, one-damper case is that the latter may be expressed as a linear operator, but this has no apparently analogous solution. We must resort to approximate methods, which are explored in the next section.

Now we consider cases with increasing numbers of springs and dampers. Figure 5 shows a numerically computed frequency response for 50 generations with one damper and various numbers of springs, ranging from one spring to 10 springs. Figure 6 shows a numerically computed frequency response for 50 generations with one spring and various numbers of dampers, ranging from one damper to 10 dampers. As with the case of two springs and one damper,

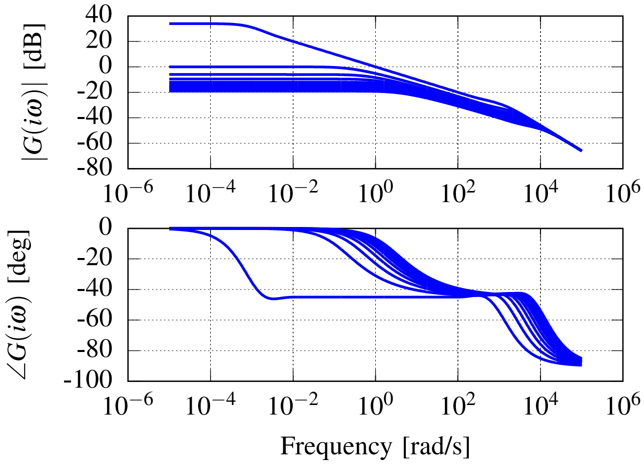

Fig. 5. Frequency response for tree system with one damper and multiple springs in each generation. The top magnitude curve is for one spring and one damper, and each lower curve in magnitude is for a system with an additional spring in each generation. The bottom phase curve is for one spring and one damper, and each higher curve is for an increased number of springs.

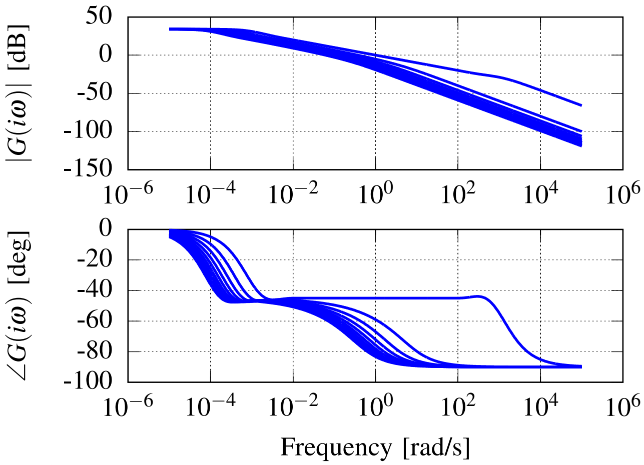

Fig. 6. Frequency response for tree system with one spring and multiple dampers in each generation. The top magnitude curve is for one spring and one damper, and each lower curve in magnitude is for a system with an additional damper in each generation. The top phase curve is for one spring and one damper, and each lower curve is for an increased number of dampers.

the operator describing the relationship between the relative displacement of the first and last generations and the force is only implicitly defined by Equation 2.

IV. Fractional-Order System Identification

We adopt a simple system identification approach. Fractional-order system identification has been studied previously [23]–[28].

A. System Identification Procedure

In order to model and control systems such as in Figure 4, 5 and 6 we have to approximate the implicitly-defined transfer function. The approach is to assume an approximate transfer function, $A_{N,\alpha}(s)$ of the form

$$A_{N,\alpha}(s) = \frac{1}{\sum_{j=0}^{N} a_j s^{j\alpha}}$$

where α is the increment in order, which we typically take to be fractional. For example if $N = 13$ and $\alpha = 1/6$ then

$$A_{13,1/6}(s) = \frac{1}{a_{13}s^{13/6} + a_{12}s^2 + a_{11}s^{11/6} + \cdots + a_1 s^{1/6} + a_0}.$$

To determine the coefficients, a_j we select a set of frequencies, $\Omega = \{\omega_0, \omega_1, \ldots, \omega_M\}$ and the objective functions

$$\phi_1(a) = \sum_{k=0}^{M} \|G(i\omega_k) - A_{N,\alpha}(i\omega_k)\|$$

$$\phi_2(a) = \sum_{k=0}^{M} \frac{\|G(i\omega_k) - A_{N,\alpha}(i\omega_k)\|}{\|G(i\omega_k)\|}$$

where the difference between the two is that the second is simply normalized by the magnitude of G. The optimization formulation is then

$$\min_a \phi(a) \quad \text{subject to} \quad a_i > 0 \quad \text{for} \quad i \in \{0, N\}$$

B. Identification of Various Systems

Taking $k = b = 1$, 50 generations, the bifurcating tree structure from Figure 1 ($n = m = 1$), $N = 13$, $\alpha = 1/6$, using $\phi_1(a)$ as the objective function, and taking 20 frequencies to be evenly logarithmically spaced[1] from $\omega_0 = 10^{-2}$ to $\omega = 10^2$ we obtain to three digits of precision the coefficient values listed in the second column ($n = 1$) in Table I.[2] When the tree has one spring and one damper in each generation, the system is very predominantly 1/2-order. However, when there is one damper and multiple springs ($n > 1$) , the system changes order and is of the form

$$A_{N,\alpha}(s) = \frac{1}{a_4 s^{2/3} + a_0}.$$

The transition from a purely 1/2-order system to 2/3-order is illustrated graphically in Figure 7.

Now we compare the frequency response of the 50-generation system, the frequency response of the self-similar, infinite-generation system and the identified system. Figure 8 illustrates all three transfer functions. Clearly, the identified system (red) matches the 50-generation system (blue) well over the range of frequencies considered in the identification ($\omega \in [10^{-2}, 10^2]$), but deviates for higher frequencies. This is somewhat exaggerated by the logarithmic nature of the plot; however, if the *order* of the system is of primary interest, then an objective function normalized by the magnitude of the transfer function to give a percentage error is desirable.

[1]That is `logspace(-2,2,13)`.

[2]The results in this paper we obtained using the Octave `sqp()` function with random initial values for $a_i \in (0, 1)$. Space limitations prevent including all cases, but for this one and all the others presented in this paper, we also studied the system identified for various order steps (α), frequency ranges, number of terms, etc., and they were all qualitatively the same. Hence, none of the results depend on the exact parameters chosen for the system identification.

TABLE I

IDENTIFIED FRACTIONAL-ORDER TRANSFER FUNCTION COEFFICIENTS WITH A VARIABLE NUMBER OF SPRINGS.

coef \ n	1	2	3	4	5	6	7
a_0	0.00	0.98	1.97	2.96	3.95	4.93	5.92
a_1	0.00	0.00	0.00	0.00	0.00	0.00	0.00
a_2	0.00	0.00	0.00	0.00	0.00	0.00	0.00
a_3	1.00	0.00	0.00	0.00	0.00	0.00	0.00
a_4	0.00	1.01	1.05	1.11	1.17	1.23	1.28
a_5	0.00	0.00	0.00	0.00	0.00	0.00	0.00
a_6	0.00	0.00	0.00	0.00	0.00	0.00	0.00
a_7	0.00	0.00	0.00	0.00	0.00	0.00	0.00
a_8	0.00	0.00	0.00	0.00	0.00	0.00	0.00
a_9	0.00	0.00	0.00	0.00	0.00	0.00	0.00
a_{10}	0.00	0.00	0.00	0.00	0.00	0.00	0.00
a_{11}	0.00	0.00	0.00	0.00	0.00	0.00	0.00
a_{12}	0.00	0.00	0.00	0.00	0.00	0.00	0.00
a_{13}	0.00	0.00	0.00	0.00	0.00	0.00	0.00

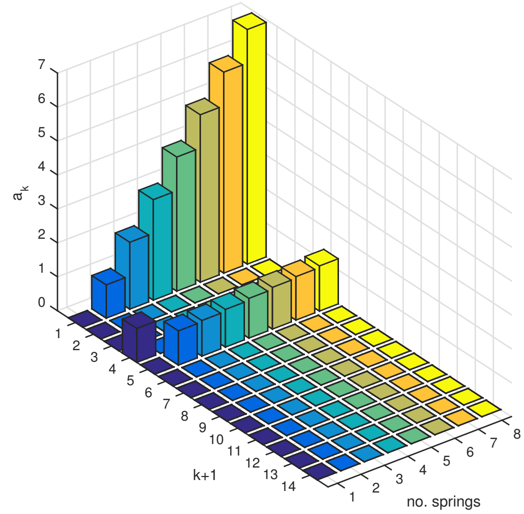

Fig. 7. Identified fractional-order transfer function coefficients with one damper and various numbers of springs. For one spring the system is purely 1/2-order ($a_3 \neq 0$) and for more springs it is 2/3-order plus a constant.

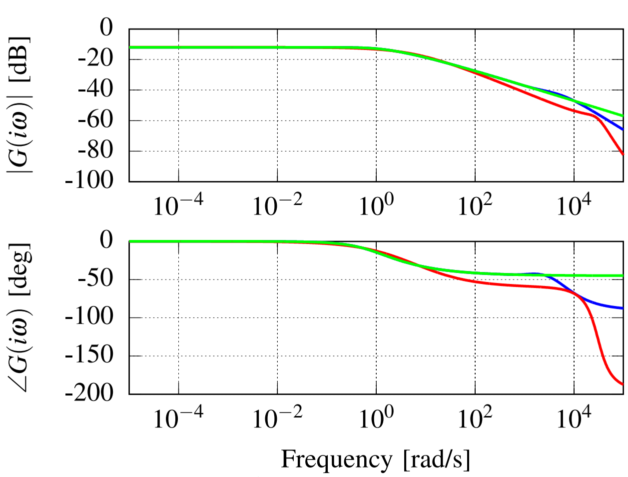

Fig. 8. Comparison of 50-generation transfer function (blue) with infinite-generation self-similar limit (green) with identified system (red). This is for five springs and one damper. Other combinations are qualitatively similar with respect to how well the identified transfer function matches the actual.

TABLE II

IDENTIFIED FRACTIONAL-ORDER TRANSFER FUNCTION COEFFICIENTS WITH VARIABLE NUMBER OF DAMPERS.

coef \ m	1	2	3	4	5	6	7
a_0	0.00	0.02	0.02	0.02	0.02	0.02	0.02
a_1	0.00	0.00	0.00	0.00	0.00	0.00	0.00
a_2	0.00	0.00	0.00	0.00	0.00	0.00	0.00
a_3	1.00	0.00	0.00	0.00	0.00	0.00	0.00
a_4	0.00	0.15	0.24	0.38	0.56	0.77	1.01
a_5	0.00	8.43	11.7	14.6	17.2	19.5	21.6
a_6	0.00	0.00	0.00	0.00	0.00	0.00	0.00
a_7	0.00	0.00	0.00	0.00	0.00	0.00	0.00
a_8	0.00	0.00	0.00	0.00	0.00	0.00	0.00
a_9	0.00	0.00	0.00	0.00	0.00	0.00	0.00
a_{10}	0.00	0.00	0.00	0.00	0.00	0.00	0.00
a_{11}	0.00	0.00	0.00	0.00	0.00	0.00	0.00
a_{12}	0.00	0.00	0.00	0.00	0.00	0.00	0.00
a_{13}	0.00	0.00	0.005	0.13	0.29	0.56	0.95

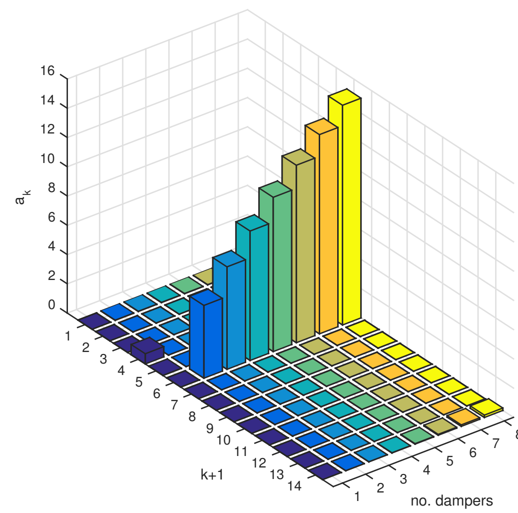

Fig. 9. Identified fractional-order transfer function coefficients with one spring and various numbers of dampers. For one spring the system is purely 1/2-order ($a_3 \neq 0$) and for more springs it is 5/6-order.

Now consider the case where there is one spring and a varying number of dampers. Table II and Figures 9 and 10 present the results. Similar to the case with a varying number of springs, there is a significant order shift from the $n = m = 1$ case to the cases where there are more dampers in each generation. However, in this case the shift is to $5/6$ order rather than $2/3$ order.

Figures 8 and 10 illustrate a deviation from the actual system at higher frequencies. This is not surprising given that the objective function is not normalized with respect to the magnitude of $G(s)$. Using $\phi_2(s)$ instead of $\phi_1(s)$ normalizes the error and should have the identified system match the *order* of the actual system better. Whether this is important is problem-dependent, but because this investigation is primarily considering the fractional-order nature of the implicitly defined operator, these results are included as well.

Tables III and IV and Figures 11 - 14 illustrate the normalized identifications for multiple springs and dampers, respectively. As is apparent, a normalized objective function

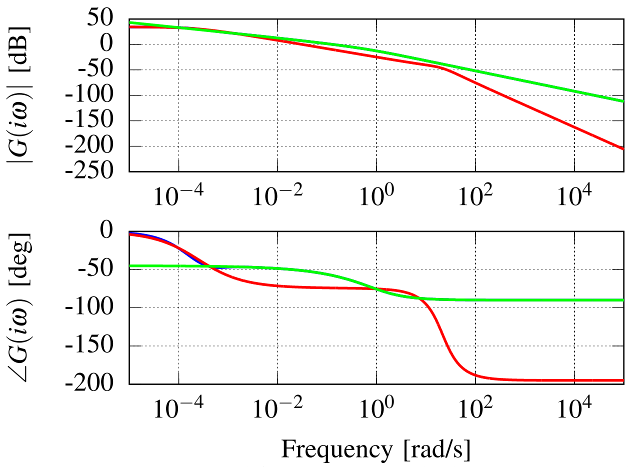

Fig. 10. Comparison of 50-generation transfer function (blue) with infinite-generation self-similar limit (green) with identified system (red) for five dampers and one spring.

TABLE III

Identified fractional-order transfer function coefficients with a variable number of springs.

coef \ n	1	2	3	4	5	6	7
a_0	0.00	0.94	1.13	2.59	3.93	4.77	2.82
a_1	0.00	0.00	1.25	0.09	0.00	0.00	3.31
a_2	0.00	0.00	0.00	0.20	0.00	0.00	0.66
a_3	0.71	0.82	0.51	1.07	1.25	1.27	0.57
a_4	0.00	0.03	0.18	0.05	0.06	0.10	0.19
a_5	0.00	0.00	0.00	0.00	0.00	0.00	0.01
a_6	0.00	0.00	0.00	0.00	0.00	0.00	0.00
a_7	0.00	0.00	0.00	0.00	0.00	0.00	0.00
a_8	0.00	0.00	0.00	0.00	0.00	0.00	0.00
a_9	0.00	0.00	0.00	0.00	0.00	0.00	0.00
a_{10}	0.00	0.00	0.00	0.00	0.00	0.00	0.00
a_{11}	0.00	0.00	0.00	0.00	0.00	0.00	0.00
a_{12}	0.00	0.00	0.00	0.00	0.00	0.00	0.00
a_{13}	0.00	0.00	0.00	0.00	0.00	0.00	0.00

results in a different order; namely, 1/2-order, and the Bode plots comparing the frequency responses indicate a much better match in the logarithmic sense of the Bode plots.

V. Conclusions and Future Work

This paper considered the dynamics of various tree-like configurations of interconnected mechanical systems. In particular, we studied systems that, in the limit of an infinite number of generations, is characterized by self-similarity. Only in one case, one spring and one damper added at each generation, can the relationship between the displacement of the network and its force be expressed in a form that has a time domain linear operator expressions (a pure fractional derivative). When we consider different combinations with more springs or dampers, we can not determine a time domain representation that is exactly a fractional derivative. This paper presented a fractional-order system identification procedure which shows that even though these cases do not have an exact representation in terms of a fractional derivative, they are still primarily fractional-order in nature.

TABLE IV

Identified fractional-order transfer function coefficients with variable number of dampers.

coef \ m	1	2	3	4	5	6	7
a_0	0.00	0.02	0.02	0.01	0.01	0.01	0.01
a_1	0.00	0.00	0.00	0.00	0.00	0.00	0.00
a_2	0.00	0.00	0.00	0.00	0.00	0.00	0.00
a_3	0.71	0.71	0.88	1.03	1.17	1.31	1.43
a_4	0.00	0.20	0.23	0.23	0.22	0.20	0.18
a_5	0.00	0.00	0.00	0.00	0.00	0.00	0.00
a_6	0.00	0.35	0.76	1.19	1.63	2.07	2.51
a_7	0.00	0.03	0.05	0.06	0.08	0.09	0.10
a_8	0.00	0.00	0.00	0.00	0.00	0.00	0.00
a_9	0.00	0.00	0.00	0.00	0.00	0.00	0.00
a_{10}	0.00	0.00	0.00	0.00	0.00	0.00	0.00
a_{11}	0.00	0.00	0.00	0.00	0.00	0.00	0.00
a_{12}	0.00	0.00	0.00	0.00	0.00	0.00	0.00
a_{13}	0.00	0.00	0.00	0.00	0.00	0.00	0.00

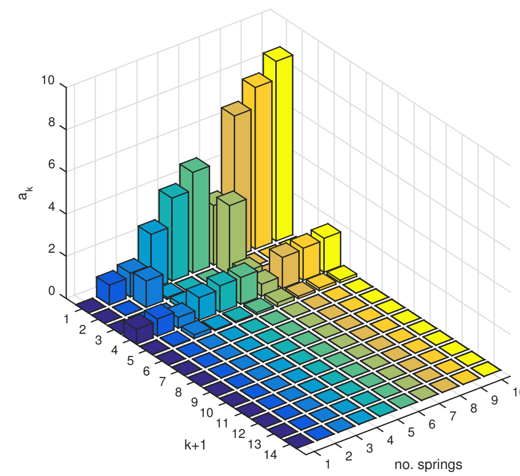

Fig. 11. Identified fractional-order transfer function coefficients with one damper and various numbers of springs with a normalized objective function. Unlike the results using ϕ_1, the system retains its 1/2-order nature, but with the addition of several other order terms.

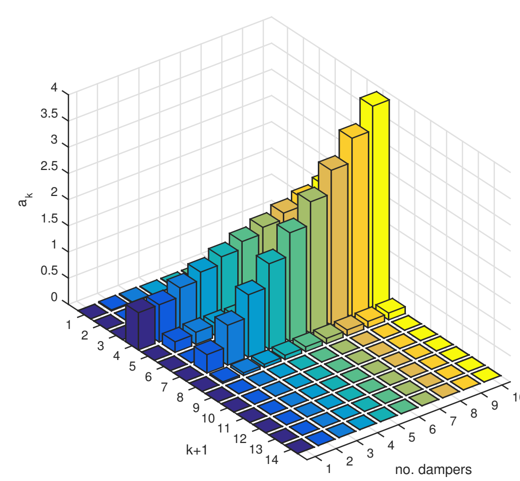

Fig. 12. Identified fractional-order transfer function coefficients with one spring and various numbers of dampers with normalized objective function. Unlike the results using ϕ_1, the system retains its 1/2-order nature, but with the addition of several other order terms.

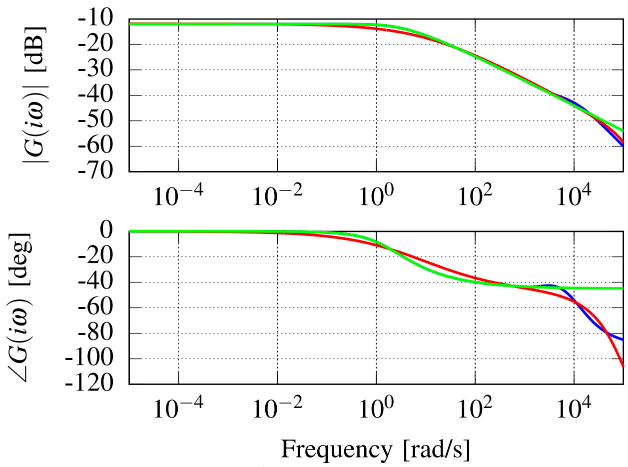

Fig. 13. Comparison of 50-generation transfer function (blue) with infinite-generation self-similar limit (green) with identified system (red) for five springs and one damper.

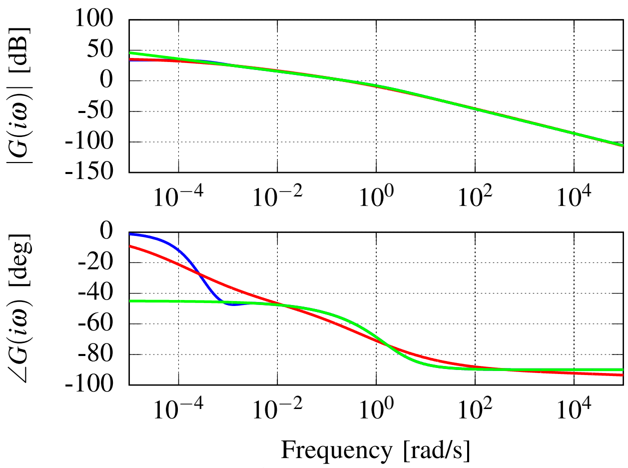

Fig. 14. Comparison of 50-generation transfer function (blue) with infinite-generation self-similar limit (green) with identified system (red). This is for five dampers and one spring. These plots illustrate a better system identification when the logarithmic nature of the Bode plot is considered.

This is important to understand because good models for systems are important in control analysis and design.

Future work includes comparing the fractional-order system model with other approaches and also validating the model with various time domain responses. Also work to determine methods for computing solutions to the operator quadratic equation, such as a generalized form of Newton's method and using homotopy methods, will be considered.

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